Individual Work 5.4: Growing Up (or Down)?

In this set of exercises, you practice graphing exponential equations described by both recursive and closed-form equations. You also practice finding the growth factor from some given conditions about the growth. Finally, you practice solving exponential equations and extend translations to exponential graphs.

1. For each of the following sequences, graph A_n versus n.

   a) $A_n = 1.15 \cdot A_{n-1}$, $A_0 = 50$

   b) $A_n = 12(1.25)_n$

2. Graph the following exponential functions.

   a) $y = 30(0.80)^x$

   b) $P(t) = 25(1.20)^t$

For quantities that change exponentially, the term **exponential growth** is used when it is *increasing*. When it is *decreasing*, the term **exponential decay** is used instead.

3. The following exponential equations were displayed using a computer drawing utility: $y = 4(1.25)^x$; $y = 10(1.25)^x$; $y = 12(1.10)^x$; and $y = 10(0.75)^x$. The graphs are shown in **Figure 5.21**.

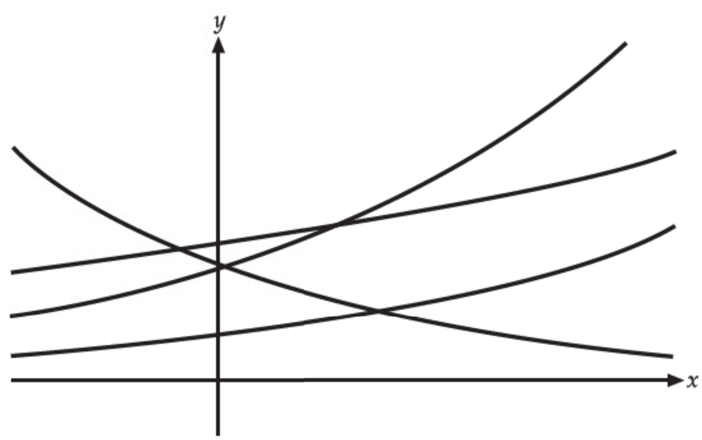

Figure 5.21.

   Unfortunately, the curves were not labeled and the axes don't show any scale. Explain how you can identify the graph of each equation shown.

Examination Copy © COMAP Inc. Not for Resale

4. Suppose two brothers are sharing a soft drink, passing the cup back and forth between them. They agree that each time they have the cup they will drink half of what is left, and then pass it to the other brother. Assume they started with a 32-ounce drink.

   a) Recursively express the amount of drink left in the cup after someone drinks.

   b) How much is left in the cup after they pass it back and forth four times (after five drinks)?

   c) Make a graph of the amount of drink left in the cup versus the number of drinks taken. Is the amount left in the cup changing linearly, exponentially, or in some other way? Explain.

Sometimes, you know that a situation is growing exponentially, but you only have some specific values. From that you must determine the growth factor (or the relative rate). The next set of exercises looks at this type of problem and how to handle it.

5. Suppose a multiplicative growth process starts out with an initial value $A_0 = 25$, and grows to a value of 49 after 2 years.

   a) Draw a two-step arrow diagram to illustrate the following situation:

      Start with the value 25, multiply it by the yearly growth factor, b, then multiply by the growth factor again, and get an answer of 49.

      Why do you multiply by the growth factor *twice*?

   b) Using the arrow diagram, write an equation that describes this same situation.

   c) Take the first step toward solving your equation. (What do you do first?) Now what does the number on the other side of the equation from the variable term represent?

   d) Take the second step in solving your equation. What solution do you get? Does that answer match the situation described in the problem?

6. The following situations all involve exponential growth. Use the technique that was developed in Question 5 to find the growth factor for each situation. Then express your answer as a relative rate. Round your answers to 3 decimal places.

   a) $A_0 = 100$, $A_2 = 120$

   b) $y(2) = 28$, $y(4) = 32$

   c) $P(0) = 1.6$, $P(3) = 2.4$

7. For each of the exponential graphs in **Figures 5.22–5.25**, identify the values of r and a as well as possible. Think of axis labels that seem reasonable for each graph. (If necessary, estimate values from the graph.)

a)

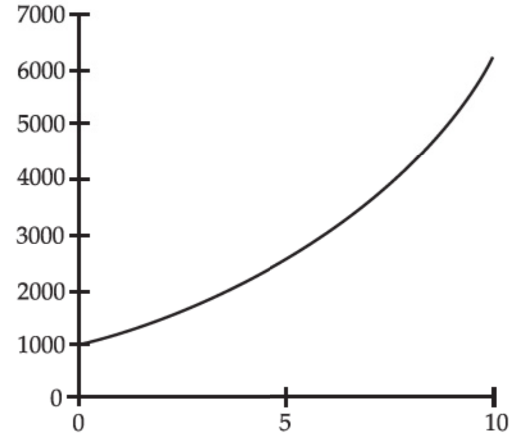

Figure 5.22.
An exponential graph.

b)

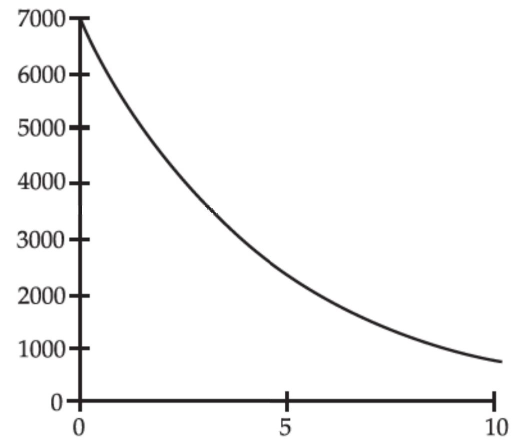

Figure 5.23.
An exponential graph.

c)

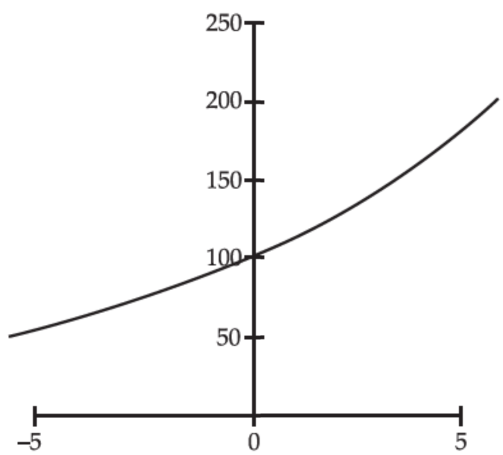

Figure 5.24.
An exponential graph.

d)

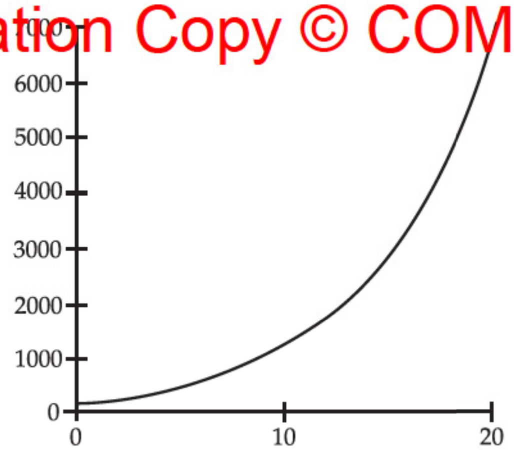

Figure 5.25.
An exponential graph.

8. The exponential equation $A_t = A_0 \cdot b^t$ has four quantities in it, where b is the growth factor per period. For each of the following, three of the quantities are provided. Describe how to solve for the missing item, and then find its value.

   a) $A_0 = \$800$; $b = 1.08$; $t = 5.2$ years

   b) $A_t = \$1200$; $b = 1.12$; $t = 3.8$ years

   c) $A_0 = \$1000$; $A_t = \$1100$; $t = 5$ years

9. The 2000 Census determined the official population figures used to allocate representatives to Congress. As of April 1, 2000, the population of the United States was around 281,422,000. That was a 13.2% increase over the 1990 census figure.

   a) What was the yearly percentage increase in the population?

   b) At that same rate of increase, what would be the population of the United States on April 1, 2003?

   c) If the population were to increase at a yearly rate of 1.00% instead, how much of an effect would that have on the population estimate from part (b)?

Banking is one context that uses exponential equations regularly. For instance, compound interest is stated as a yearly percentage, but actually earned over a shorter time interval. Sometimes, that **period** is stated as a fractional part of a year, like 1/4 year. More likely, it is expressed as the **number of compounds per year**, like 4 times per year. That fractional part of the yearly percentage becomes the relative rate for calculating the interest according to an exponential equation. The account balance is calculated by the formula:

$$A = A_0\left(1+\frac{r}{n}\right)^{n \cdot t}$$

TAKE NOTE

The compound interest formula looks complicated, but it is simply an equation of the form $y = a \cdot bx$. The r/n calculates the relative rate per *compound period*. The growth factor, b, for that time period is equal to $1 + r$, so the expression inside the parentheses is just the growth factor per compound period. The exponent is adjusted to account for the fact that the time intervals are actually fractional parts of years. (This is a nice use of scale-change transformations!)

where A_0 is the initial amount deposited, r the yearly percentage rate, n the number of times the interest is compounded per year, and t the number of years.

10. Shania put $500 in a bank account that earned interest at 6% per year, compounded quarterly.

    a) Calculate the balance in the bank account at the end of the first quarter. What was the growth factor over that first compound period?

    b) What would the account balance be at the end of the second, third, and fourth quarters?

    c) What is the actual growth factor over the entire first year? Convert your answer to a yearly percent increase. (Note: the yearly relative rate is also called the **effective yield** for that investment.)

    d) How could you calculate the effective yield from the annual rate without first calculating the balance at the end of the first year?

11. For each of the following banking situations, which describe an account earning compound interest, calculate the balance at the end of the specified time period.

    a) A_0 = $1000; r = 8% compounded 6 times per year; t = 5 years.

    b) A_0 = $800; r = 5% compounded monthly; t = 12 years.

12. Suppose a car costs $10,000 and loses 20% of its value every year due to depreciation. What would the value of the car be after 6 years?

13. The population of steelhead trout in California's rivers was estimated to be 3600 in 1980. The total number of these fish was going down at the rate of 2% per year, due to pollution and over-development on their spawning grounds.

    a) Based upon this model, what would the population of steelhead in California be in the year 2000?

    b) It is likely that the decay rate was rounded off to 2%. If so, it could have been any value between 1.5% and 2.5%. What would be the highest and lowest estimates for the fish populations, based on these facts?

14. It takes an income of $40,000 per year to live comfortably in retirement right now. Inflation causes the cost of living to go up by around 2.5% each year. What income per year would it take to live comfortably, if you retire 25 years from now?

In Individual Work 5.2, the idea of translating graphs of linear processes was revisited. Shift transformations are a process that can be applied to

Examination Copy © COMAP Inc. Not for Resale

any type of equation. In this next set of exercises, you explore how to apply translations to graphs of exponential processes.

15. Start with a simple exponential equation, $y = 10(1.5)^x$. You would like to shift it two units to the right, and you want to know what equation will describe this new pattern.

   a) The "hard" way to find the equation would be to consider the value 10 as being $y(2)$. You could then use the growth process in reverse to work back to what the new value must be at $y = 0$. What equation do you get?

   b) Verify your work in part (a). Make a table of x- and y-values, using the original equation. Then shift the table values so that each y-coordinate gets an x-coordinate two units larger than before. Make a scatterplot of your new data points and your graph from part (a) on the same set of axes. Does your graph go through the data points?

You've earned the right to use an "easy" way. Recall from Individual Work 5.2 that graphs can be translated simply by altering the equation. You found parametric equations a useful tool for doing that.

   c) Replace the variable x with the expression that describes what the new x-coordinates are. What equation do you get?

   d) Graph your answer to part (a) with that of part (c) on the same set of axes. Do the graphs look the same? (Are the equations equivalent?)

16. Translate the graphs of the given equations by the specified amount. Graph the original equation and the one that includes the "shift" to verify that it works correctly.

   a) $y = 20(1.25)^x$; shift three units to the right.

   b) $A = 50 \cdot (0.90)^t$; shift it one unit to the left.

   c) $P_{t+1} = 1.40 \cdot P_t$, $P_0 = 58$. Shift two units to the right.

17. Every exponential growth graph has a related decay graph, which can be found without much work. Consider the growth that is described by the equation $y = 100(1.25)^x$.

   a) Graph this function on the window $[-10, 10] \times [0, 1000]$.

Think of making a decay process by reflecting your closed-form graph across the y-axis, as shown in **Figure 5.26**.

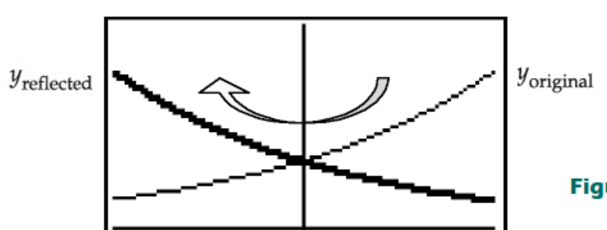

Figure 5.26.

Examination Copy © COMAP Inc. Not for Resale

The decay process will have to have an equation that looks like: $y = 100 \cdot b^x$. (Why?) The growth factor for the decay process can be found by calculating the growth factor for the original equation "in reverse." For example, find the growth factor in going from $y(1)$ to $y(0)$ in the original equation.

TAKE NOTE Think of exponential decay as being exponential growth going back in time (like rewinding a video). The act of reversing time (switching the orientation on the horizontal axis) reflects the graph across the vertical axis.

b) What will be the equation for the decay process?

c) Graph the decay function on the same window. Verify that it really is the reflection of the original graph about the y-axis by checking to see that $y_{\text{original}}(2) = y_{\text{reflected}}(-2)$.

d) Graph the function $y = 100(1.25)^{-x}$ on the same window. Describe what you see, and explain why that happens.

For any exponential equation, $y = a \cdot b_x$, the graph can be reflected across the y-axis by changing the equation in any of the following ways:

$y = a \cdot (b^{-1})^x$ $\quad y = a \cdot (1/b)^x$ $\quad y = a \cdot b^{-x}$