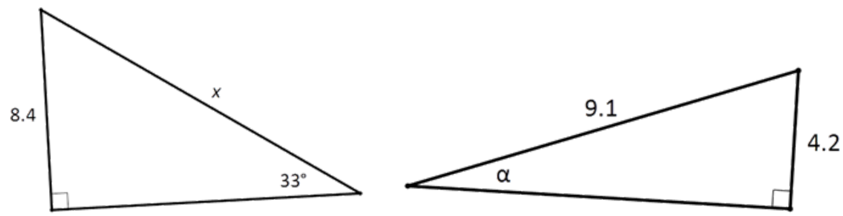
6.6 Laws of Sines & Cosines

Digging Deeper into Triangle Trig

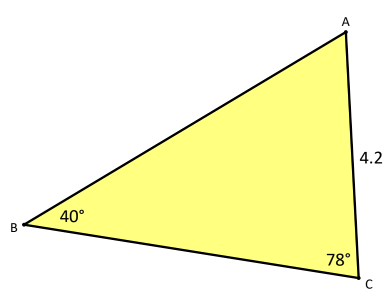
In the previous section, you used trigonometric ratios to solve right triangles. The trigonometric functions can also be used to solve *oblique triangles* (triangles with no right angles). We do this by using the Law of Sines and Law of Cosines.



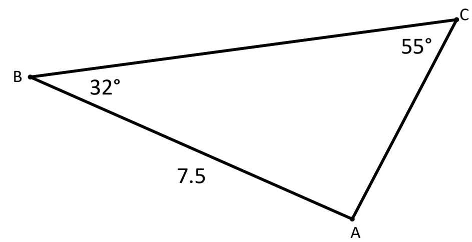
1. Find the value of x in the triangle on the left. (Figure above)

2. Find the value of α in the triangle on the right. (Figure above)

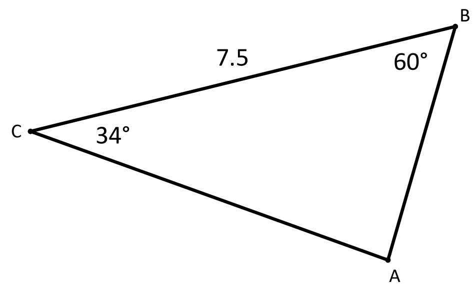
3. Find all of the measurements for the triangle below.



4. Find the length of side AC in the triangle below.



5. A hiker at point C is 7.5 kilometers from a hiker at point B; a third hiker is at point A. Use the angles shown in the diagram above to determine the distance between the hikers at points C and A.



The Law of Sines says that in any triangle, the lengths of the sides are proportional to the sines of the corresponding opposite angles.

For triangle ABC defined above:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Solve the following triangle using the Law of Sines.

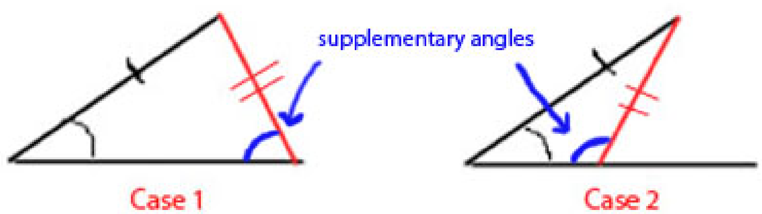
6. Two sides of a triangle have lengths 10.4 and 6.4. The angle opposite 6.4 is 36°. What could the angle opposite 10.4 be?

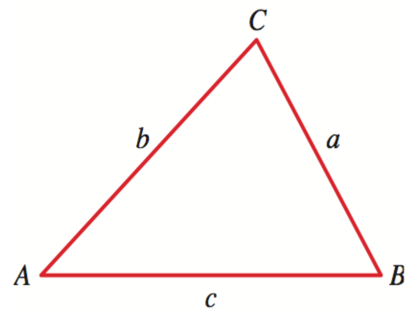
7. Two sides of a triangle have lengths 9.6 and 11.1. The angle opposite 9.6 is 59°. What could the angle opposite 11.1 be?

II. The Ambiguous Case

The so-called ambiguous case arises from the fact that an acute angle and an obtuse angle have the same sine. If we had to solve $\sin x = \frac{\sqrt{2}}{2}$, we would have $x = 45°$ or $x = 135°$.

For this reason, SSA triangles do not define necessarily define congruent triangles, and are sometimes referred to as *the ambiguous case*. The figure below illustrates this.





EXAMPLE: Triangle ABC (not drawn to scale) has $\angle A = 30^\circ$, $a = 1.5$, and $b = 2$. Find and sketch all possible triangles that satisfy these properties.

SOLUTION: Use the Law of Sines to find angle B.

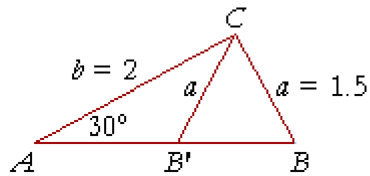
$$\frac{\sin B}{\sin 30^\circ} = \frac{2}{1.5}$$

$$\frac{\sin B}{.5} = \frac{2}{1.5}$$

$$\sin B = \frac{2}{3} \approx .6667$$

$$\sin^{-1}\frac{2}{3} = 42^\circ$$

So, one possible solution for angle B is 42°. But the sine of an angle is equal to the sine of its supplement. That is, .666 is also the sine of $180^\circ - 42^\circ = 138^\circ$. Now, let's draw a picture of the two possible triangles.



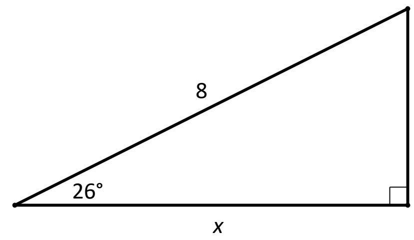
This problem has two solutions. Not only is angle CBA a solution, but so is angle $CB'A$, which is the supplement of angle CBA. Now, your turn!

8. Triangle ABC has $\angle A = 43.1^\circ$, $a = 186.2$, and $b = 248.6$. Find and sketch all possible triangles that satisfy these properties.

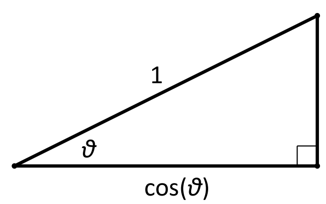
9. Triangle ABC has $\angle A = 42^\circ$, $a = 70$, and $b = 122$. Find and sketch all possible triangles that satisfy these properties.

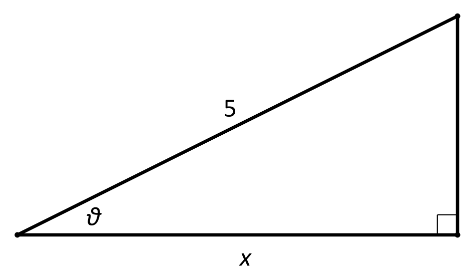
III. The Law of Cosines

10. Find the value of x in the triangle below.

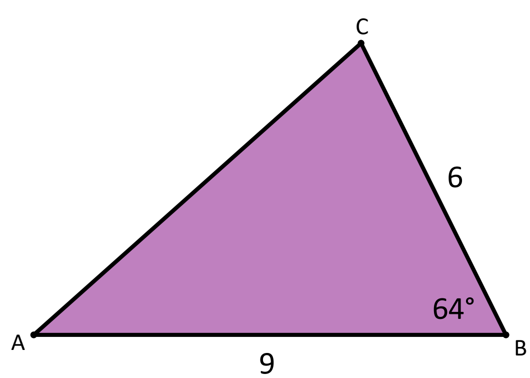


11. Explain how the figures below are related. Then, describe x in terms of ϑ.

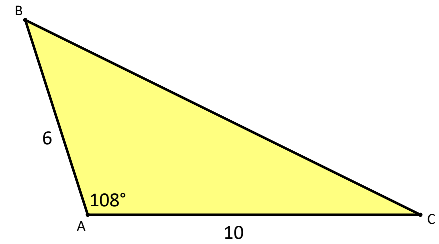




12. Find the length of side $\overline{AC}$ in the triangle below.



13. Points B and C are located at the edges of a large body of water. Point A is 6 km from point B and 10 km from point C. The angle formed between segments $\overline{BA}$ and $\overline{AC}$ is 108°. How far apart are points B and C?



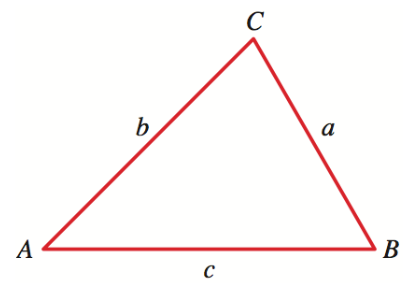
The Law of Sines cannot be used directly to solve triangles if we know two sides and the angle between them, or if we know all three sides (SAS or SSS). In these two cases, we use the Law of Cosines instead.

For triangle ABC, the **Law of Cosines** states that

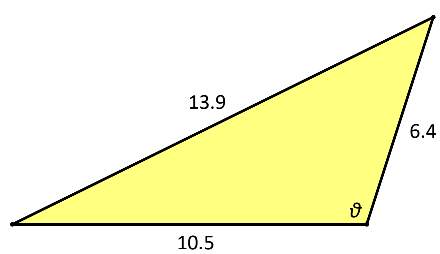
$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

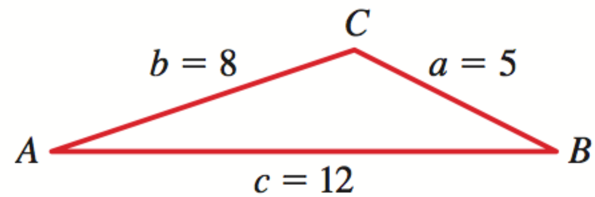
$$c^2 = a^2 + b^2 - 2ab\cos C$$



14. Use the law of cosines to find the value of ϑ in the triangle below.



15. The sides of a triangle are a = 5, b = 8, and c = 12. Find the angles of the triangle.



16. A tunnel is to be built through a mountain. To estimate the length of the tunnel, a surveyor makes the measurements shown below. Use the surveyor's data to approximate the length of the tunnel.