6.5 Overtones

Digging Deeper into the Math of Music

So far, you have spent a lot of time analyzing music through trigonometry. You should be comfortable, by now, representing sounds by analyzing their frequencies. One problem, however, is that sounds in real life are often not so simple.

Let's start with an easier problem. When you strike a tuning fork, you produce a **pure tone**. Pure tones are the most basic building blocks of sound, and can be represented as a single trigonometric function.

1. Suppose we strike a tuning fork that vibrates with a frequency of 264Hz and amplitude of 0.002 inches. This will produce a pure C note. Find the sine equation that models this note.

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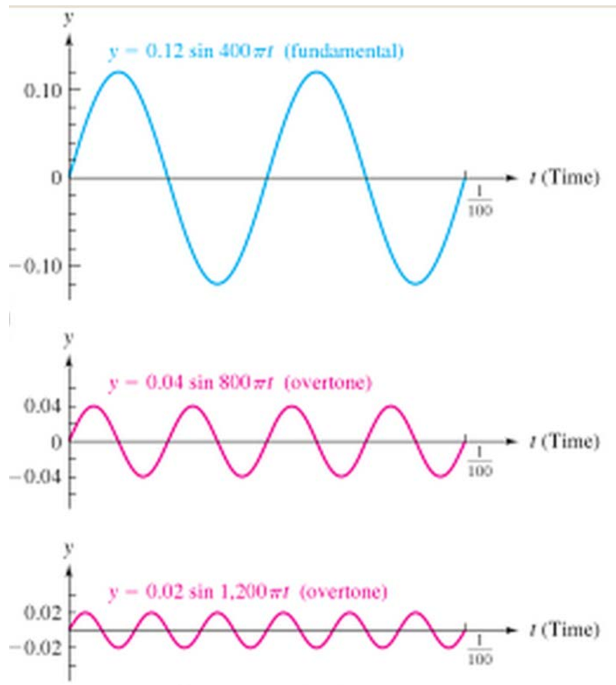
II. Harmonic Tones

When you listen to music, you do not typically hear a lot of tuning forks. In order to create more complex sounds, you need to combine a number of pure tones. You have probably noticed that different musical instruments sound different even if they are playing the same note. This is because they have different combinations of pure tones.

A **harmonic tone** is the sum of pure tones, in which the frequency of each pure tone is a multiple of the tone with the lowest frequency (called the fundamental tone). Any frequency above the fundamental tone is called an **overtone**. Any frequency above the fundamental tone that is an integer multiple of the fundamental tone is called a harmonic. For instance, if we play the A note on the piano above middle C, also called A440, 440Hz is the frequency of the first (fundamental) tone, but there is also a second harmonic (880Hz), a third harmonic (1320Hz), and so on. Each of these harmonics has varying amplitudes. Mathematically, we can represent harmonic tones through the addition of many trigonometric functions.

3. Suppose a guitar string is plucked that that vibrates with a fundamental frequency of 200Hz. List frequencies of the next two harmonics.

The graphs below represent the fundamental frequency of the guitar string plus its first two overtones. Notice how the amplitude changes in the overtones compared to the fundamental tone.



4. Using your graphing calculator, graph $y = 0.12\sin(400\pi t) + 0.04\sin(800\pi t) + 0.02\sin(1200\pi t)$. You may need to adjust your graphing window. This is a close approximation to the guitar note we are investigating. Sketch a picture of the graph below.

5. Based on your graph in question 4, what is the period of y? How does it compare with the period of each individual tone? What do you suspect determines the period for the harmonic tones?

6. The distinctive sound of a trumpet is due in part to the high amplitudes of its overtones. The following three functions are the first three harmonics for a trumpet playing middle C.

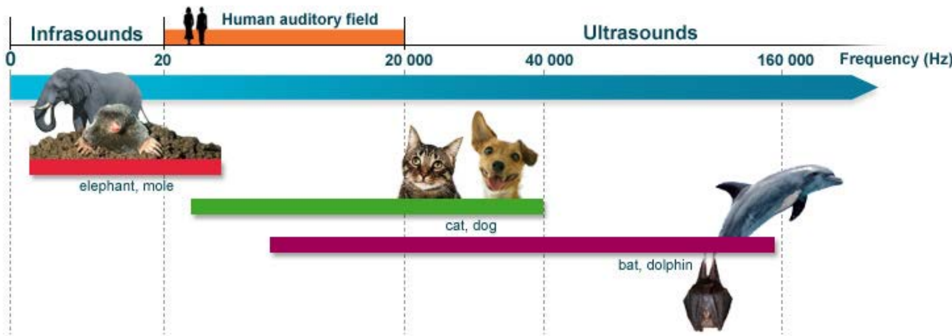
$y_1 = 0.30\sin(528\pi t)$, $y_2 = 0.28\sin(1056\pi t)$, $y_3 = 0.22\sin(2112\pi t)$

7. Use a calculator to graph separately the first three harmonics of the functions above, for $0 \leq t \leq 0.008$. Then graph the sum of the three harmonics. Sketch the graphs below. What do you notice about the period of the sum?

III. Some Interesting Harmonics Facts

Doubling the frequency (Hz) of a pitch will raise the pitch one octave. For instance, the A note we discussed earlier has a frequency of 440Hz. If we wanted to find the frequency of the next time an A occurs, we would perform the calculation $440 \times 2 = 880$Hz. After that, an A occurs at $880 \times 2 = 1760$Hz.

8. The note D_2 has the frequency 73.42Hz. Find the frequency of the D note that is one octave above and below D_2.



Humans can typically hear frequencies from 20Hz up to about 20,000Hz. Lower frequencies can potentially produce more overtones within our ranges of hearing. This makes sense because if a note has a fundamental tone of 50Hz, many of its overtones would be within our range of hearing. A very high note on a piano, however, may have a fundamental frequency of about 4000Hz. Only a couple octaves of overtones of this note would be audible to humans.

9. Let's say you play the F_3 note on a piano, which has a frequency of 174.61Hz. How many octaves of overtones would be audible to humans?

Why does Autotune sound so funny? As you now know, when one sings, there is a fundamental tone, as well as many overtones. The program Autotune isolates the fundamental tone, leaving out all of the overtones. This produces the synthetic, robotic quality that we associate with Autotune. Let's look at some graphs to help us understand this.

10. Graph the function $y = \sin(880\pi x)$ below. This is the graph of singer singing the note A440 through Autotune.

11. Next, graph the function $y = \sin(880\pi x) + \frac{\sin(1760\pi x)}{2} + \frac{\sin(2640\pi x)}{3} + \frac{\sin(3520\pi x)}{4}$

    This is an approximation of the actual sound a singer would produce singing the note A440. To make things simpler, all of the amplitudes have been set to 1. Graph this function below.

12. Compare the two graphs. What is similar about them? What is different about them?

IV. Damped Harmonic Motion

When looking at the graphs of the trigonometric functions you graphed above, you may have noticed that, because of their periodic nature, these graphs will go on to infinity in the positive and negative direction on the x axis. However, if you have ever plucked a guitar string, you know that the sound does not continue forever. It gets quieter and quieter, and eventually becomes silent. This is due to the presence of friction. The type of motion, in which the amplitude decreases over time, is called **damped harmonic motion**, and it can be represented mathematically as follows:

Damped Harmonic Motion

The equations $y = ke^{-ct}\sin\omega t$ or $y = ke^{-ct}\cos\omega t$ $(c > 0)$

model damped harmonic motion, with the constant c being the **damping constant**, k being the initial amplitude, and $2\pi/\omega$ the period.

13. A tuning fork is struck and oscillates in damped harmonic motion. The amplitude of the motion is measured, and 3 seconds later it is found that the amplitude has dropped to $\frac{1}{4}$ of this value. Find the damping constant c for this tuning fork.

14. A guitar string is pulled at point P a distance of 3 cm above its resting position. It is then released and vibrates in damped harmonic motion with a frequency of 165 cycles per second. After 2 seconds, it is observed that the amplitude of the vibration at point P is 0.6 cm.

a. Find the damping constant c.

b. Find an equation that describes the position of point P, above its rest position as a function of time. Take $t = 0$ to be the instant the string is released.