6.4 Math and Music

Turning Music Into Math

Even people that don't have a musical background are still able to listen to and appreciate music. Throughout your lifetime, you've undoubtedly heard people playing instruments and singing and said to yourself, "Wow, all these instruments and vocals sure do sound great together." Unfortunately, you've probably also heard the opposite: two people singing together or two instruments being played together that did not sound so great. So what makes some combinations of music sound beautiful, and others sound dreadful? And is there a mathematical explanation for this? (Spoiler Alert: Of course there is!)

I. Musical Background

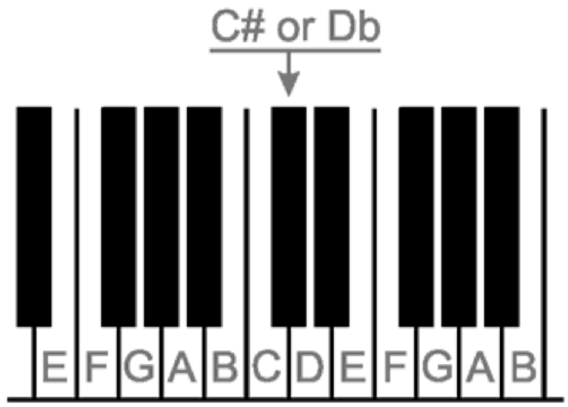
In order to make music, a musician constructs different series and patterns of pitches. A musical pitch (more commonly called a note) is determined by its frequency, which is measured in vibrations per second, or Hertz (Hz).

Surprisingly, frequency is pretty simple to find. You already know how to find the period of a trigonometric function: For a function $y = \sin(kx)$ or $y = \cos(kx)$, the period is equal to $\frac{2\pi}{k}$. Frequency is the inverse of period: the frequency, f, measured in Hertz, is $f = \frac{k}{2\pi}$.

Example: Find the period and frequency of the function $y = \sin(440\pi x)$

Did you get that the frequency is 220Hz? On a piano keyboard, this is the A note below middle C. The notes on a piano keyboard form what is called a *chromatic scale*. A chromatic scale divides the octave into its semitones. There are twelve semitones, or half steps, to an octave in the chromatic scale.

The white keys on a keyboard are A, B, C, D, E, F, and G. The black keys are named relative to their adjacent white keys. For example, the black key between the C and D keys is known as either *C sharp* (C#) or *D flat* (Db).



The frequencies of the notes that make up a chromatic scale form a geometric sequence where each term has the form ar^n. The ratio that generates the chromatic scale is $r = 2^{\frac{1}{12}}$ or $r = \sqrt[12]{2}$.

Using our knowledge that the A note below middle C has a frequency of 220Hz, we can calculate the frequencies of the terms that generate a two-octave chromatic scale.

II. Finding the Frequencies of an Octave

1. Calculate each value using the original value of a = 220 and the formula ar^n, where $r = 2^{\frac{1}{12}}$. Using a scientific calculator, this can be accomplished by the following steps:

   - Enter 220, and hit "ENTER".
   - Enter the following: $\boxed{\times}\ \boxed{2}\ \boxed{\wedge}\ \boxed{(}\ \boxed{1}\ \boxed{\div}\ \boxed{12}\ \boxed{)}$.

- Repeatedly hit "ENTER" Each time you hit "ENTER" is pressed, the value for the next row will be calculated

Round each frequency to the nearest whole number

LOWER OCTAVE	FREQUENCY (HZ)
A	220 Hz
A# or Bb	$220\left(2^{1/12}\right)^1 \approx 233$
B	$220\left(2^{1/12}\right)^2 \approx$
C	$220\left(2^{1/12}\right)^3 \approx$
C# or Db	
D	
D# or Eb	
E	
F	
F# or Gb	
G	
G# or Ab	

HIGHER OCTAVE	FREQUENCY (HZ)
A	
A# or Bb	
B	
C	
C# or Db	
D	
D# or Eb	
E	
F	
F# or Gb	
G	
G# or Ab	

2. In the table above, compare the frequencies of notes that are one octave apart. For instance, compare A in the lower octave (left

column) with A in the higher octave (right column), compare A# in the lower octave with A# in the higher octave, and so forth. How do frequencies an octave apart appear to be related? (continue answer on next page)

The sine wave related to a musical pitch has the following form, where A is the *amplitude* of the sound (or the volume, measured in decibels) and B is the frequency of the note (measured in Hz):

$$f(x) = A\sin(Bx)$$

3. Based on the frequencies in the above table, write the sine functions to represent both the low and high octaves for the C notes. (The value of A represents the volume of the note, so any value can be used. For the remainder of this activity sheet, let $A = 2$.)

Then, graph the sine function for each note on your graphing calculator, and change the viewing window to show two cycles of the curve. (To do

C Notes

Xmin: __________

Xmax: __________

Ymin: __________

Ymax: __________

Middle C
Frequency: ____________________ Period: ______________________

Lower C
Frequency: ____________________ Period: ______________________

this, set Xmin = 0, and set Xmax to twice the value of the period; remember that the period is equal to 2π divided by the frequency.) Graph the sine waves for notes in both octaves in the same viewing window. Draw the graph, and record the scale, frequency and period below

4. Repeat the same process for another set of two notes that are an octave apart.

Notes: _____

Xmin: __________

Xmax: __________

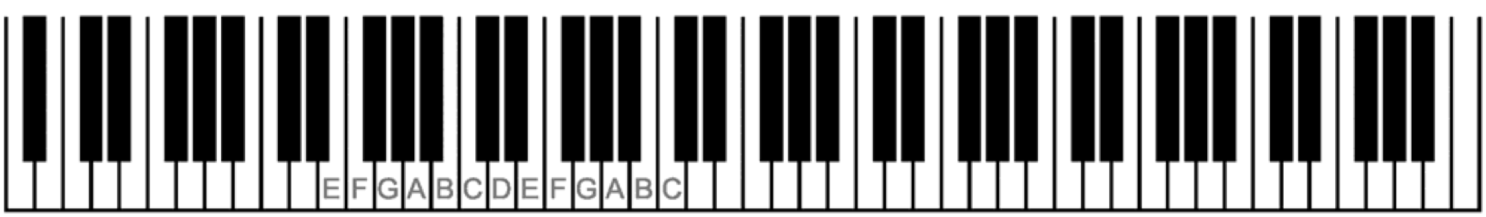
Ymin: __________

Ymax: __________

Higher Note
Frequency: ____________________ Period: ______________________

Lower Note
Frequency: ____________________ Period: ______________________

5. Based on your observations above, describe where the graphs meet. With the help of a musician in class, play C notes that are an octave apart. Describe in your own words how the notes compare. The diagram below shows how to identify keyboard notes that are an octave apart.



For Group 1

Adjacent notes in a chromatic scale are half-steps apart. A <u>diatonic</u> musical scale (also called a <u>major</u> scale) is an eight-note sequence of notes that are both half- and whole steps apart, as shown below.

The eight notes in the A-major scale are:

A |———————| **B** |———————| **C#** |———————| **D** |———————| **E** |———————| **F#** |———————| **G#** |———————| **A**

A–B	B–C#	C#–D	D–E	E–F#	F#–G#	G#–A
whole step apart	whole step apart	half step apart	whole step apart	whole step apart	whole step apart	half step apart

6. A major chord (or triad) of any scale consists of the *first, third,* and *fifth* notes of the scale. Based on the A major scale identified above, identify the notes of the A major chord, the frequencies of those notes, the associated sine function with A = 2, and the period of the sine wave.

NOTE	NAME	FREQUENCY (B)	SINE FUNCTION $f(x) = 2\sin(Bx)$	PERIOD OF SINE WAVE
First note of the A major scale	A	220 Hz		
Third note of the A major scale				
Fifth note of the A major scale				

7. When all three of the above sine waves are graphed, they intersect at the point (0,0).
   a. What are the coordinates of the second point where all three sine waves intersect? (The first point of intersection in the window should be the origin.)

      (_____________ , _____________)

   b. From the origin to the next point of intersection, record the number of cycles for each of the sine waves.

Note of the Chord	Number of Cycles
First note of the scale	
Third note of the scale	
Fifth note of the scale	

For Group 2

Adjacent notes in a chromatic scale are half-steps apart. A <u>diatonic</u> musical scale (also called a <u>major</u> scale) is an eight-note sequence of notes that are both half- and whole steps apart, as shown below.

The eight notes in the C-major scale are:

C	D	E	F	G	A	B	C
whole step apart	whole step apart	half step apart	whole step apart	whole step apart	whole step apart	half step apart	

8. A major chord (or triad) of any scale consists of the *first, third,* and *fifth* notes of the scale. Based on the C major scale identified above, identify the notes of the C major chord, the frequencies of those notes, the associated sine function with A = 2, and the period of the sine wave.

Note	Name	Frequency (B)	Sine Function $f(x) = 2\sin(Bx)$	Period of Sine Wave
First note of the A major scale	C	262 Hz		
Third note of the A major scale				
Fifth note of the A major scale				

9. When all three of the above sine waves are graphed, they intersect at the point (0,0).
   a. What are the coordinates of the second point where all three sine waves intersect? (The first point of intersection in the window should be the origin.)

      (____________ , ____________)
   b. From the origin to the next point of intersection, record the

number of cycles for each of the sine waves.

NOTE OF THE CHORD	NUMBER OF CYCLES
First note of the scale	
Third note of the scale	
Fifth note of the scale	

For Group 1 and 2

10. Compare your findings regarding the notes making up the A major scale and the notes making up the C major scale. Then, experiment with other major scales that start with other notes. Do the sine waves of the triad (the first, third, and fifth notes) compare in the same way for those scales? Summarize your findings below.

11. Experiment with other note combinations. Graph the sine wave of two notes that are half-steps apart, such as A and A#. How many cycles do you have to graph before the two curves intersect? Then, have a musician play A and A# together to hear how they sound. Would you describe the sound as harmonious or dissonant?

    Similarly, compare sine waves for three notes that are half-steps apart, such as B, C, and C#, and listen to those notes played together.

    In the space below, describe what dissonance looks like and what harmony looks like in terms of sine waves.