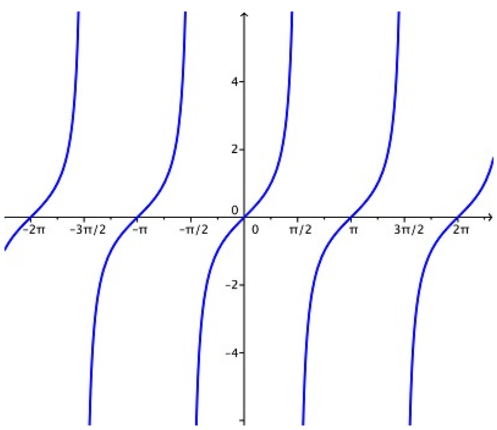
6.3 Seeing Music

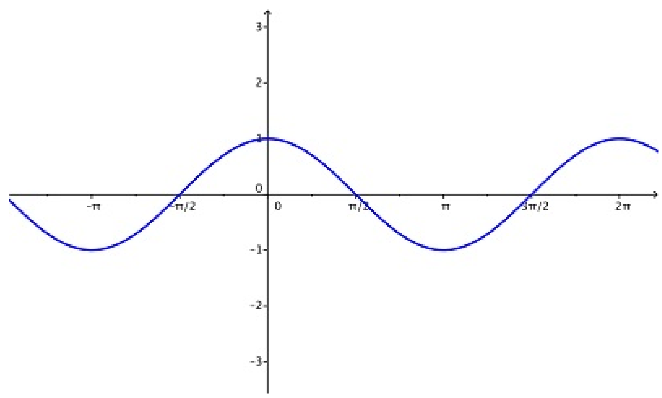
Graphs of Sine and Cosine

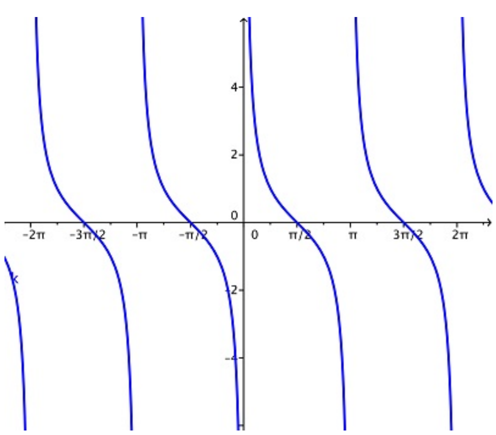
Sound can be represented visually through waves. These waves take the form of the sine and cosine graphs that you have studied. Now that you know the general shape of these graphs, let's analyze them further.

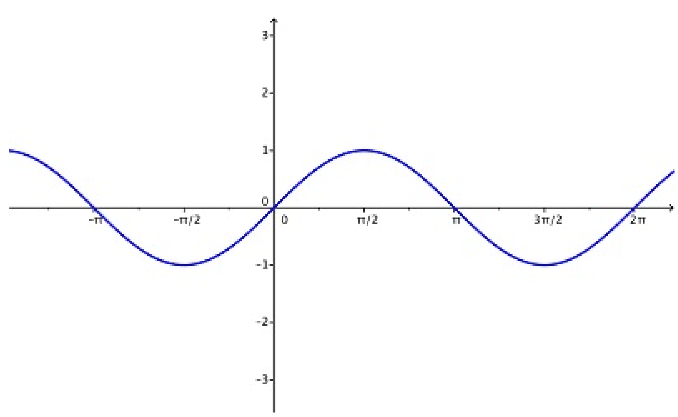
"Holi Festival of Colors Utah, United States 2013" by Steven Gerner - Flickr: Holi / Festival of Colors 2013 via Wikimedia Commons

1. The graphs below depict four trigonometric functions. Identify which of the graphs are $f(x) = \sin(x)$, $g(x) = \cos(x)$, and $h(x) = \tan(x)$. Explain how you know. Which of the graphs form the expected shape of a "sound wave?"

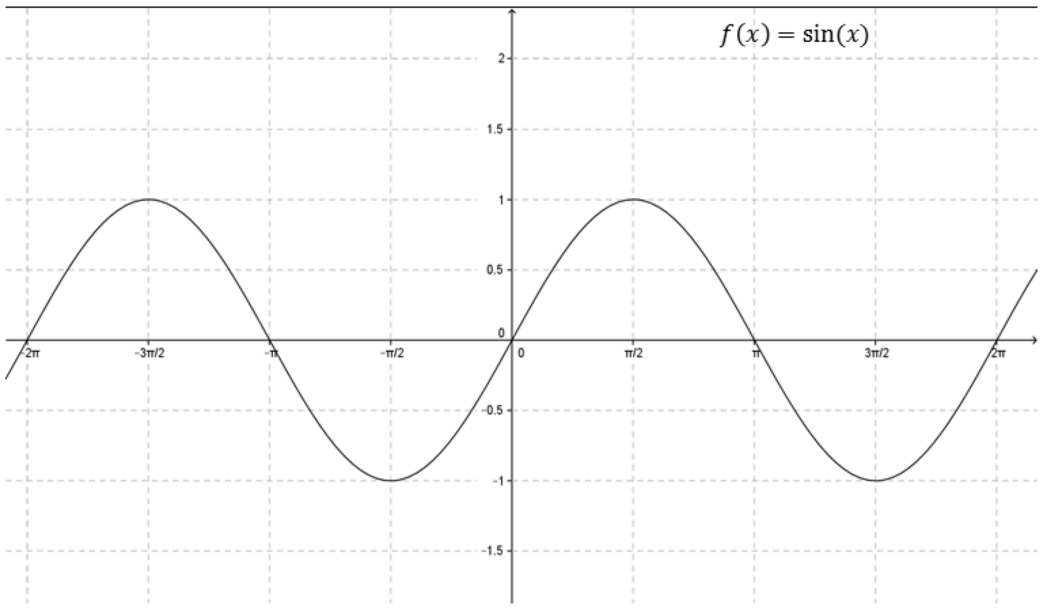


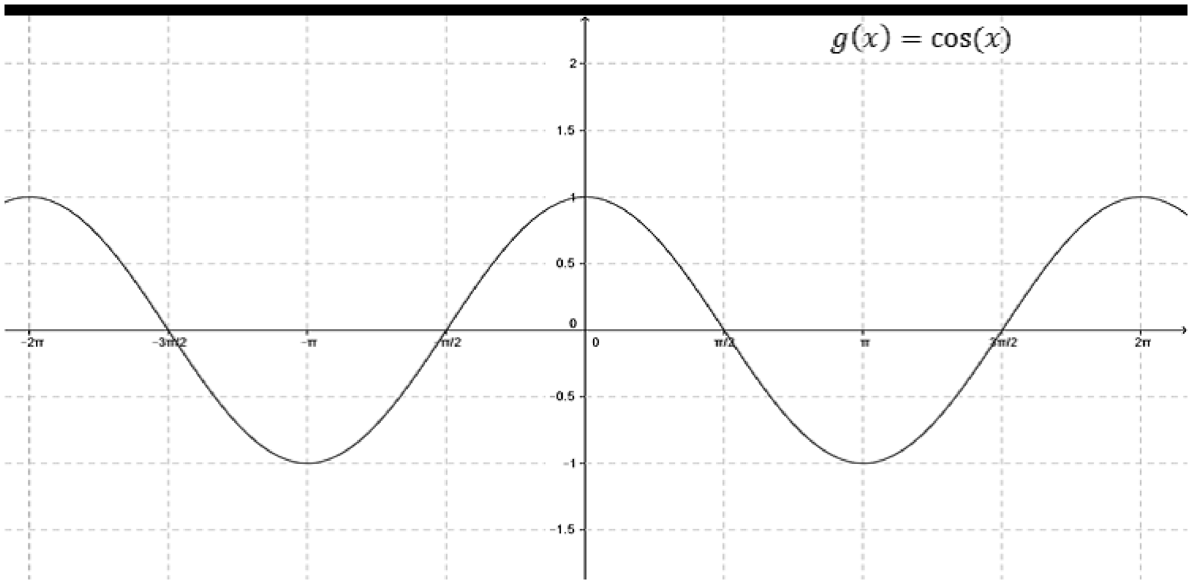






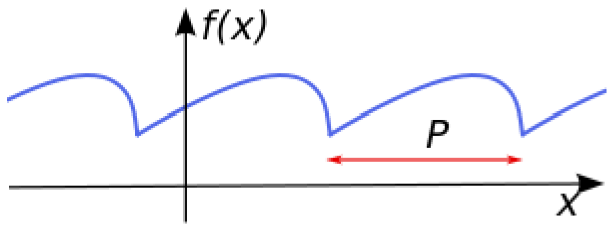
2. Look again at the graphs of the functions $f(x)=\sin(x)$ and $g(x)=\cos(x)$. Describe the symmetry of the graphs: which graph has reflection symmetry and which has rotational symmetry. Label the center of rotation and draw the line of reflection.





3. How could you translate the sine graph so it would coincide with the cosine graph. Write this as an identity. Explain how this relates to the unit circle.

II. Periodic Functions



We have established that the sine and cosine graphs will repeat forever in the positive and negative direction. Let's put a formal definition to this idea. A function is **periodic** if it repeats itself in some way forever. In math, we would say that a function $f(t)$ is periodic if there is some number p such that $f(t+p) = f(x)$ for every t.

The smallest length p that is repeated is called the **period** of f. The graph will repeat itself after every interval of length p. Each interval of length p is called a **cycle**.

4. When do the graphs of sine and cosine repeat themselves?

    a. Sine and cosine are periodic functions with a period of ______________.

    b. This means that $\sin(t + \_\_\_\_\_\_\_\_)\ = \sin t$ and $\cos(t + \_\_\_\_\_\_\_\_)\ = \cos t$

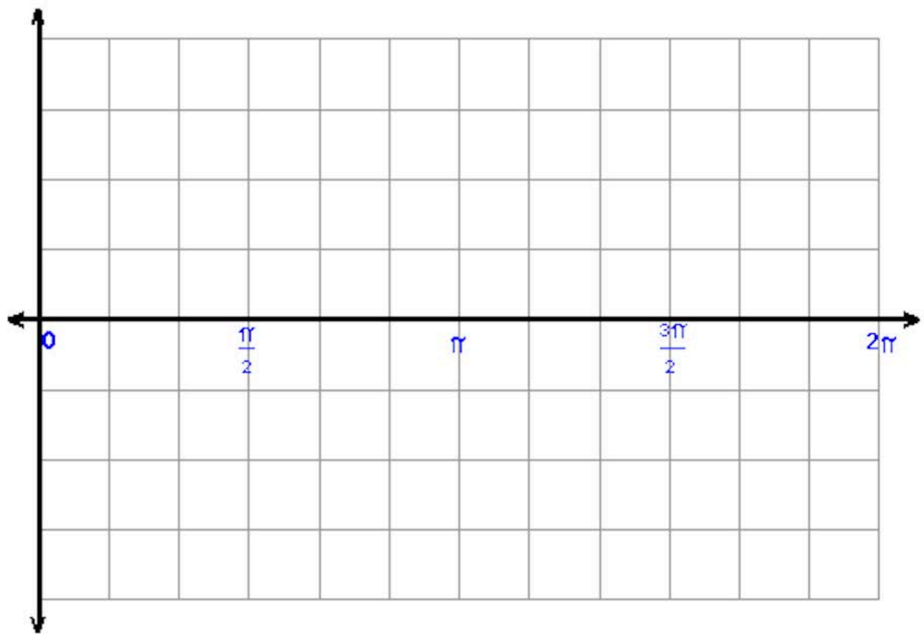
Let's conduct a test to see if this is true:

5. Choose an angle measurement (in radians) that is on a unit circle. Record it below:
    a. Your angle measurement: ____________
    b. Find the terminal point at that angle measurement.
        i. Terminal Point: _____________

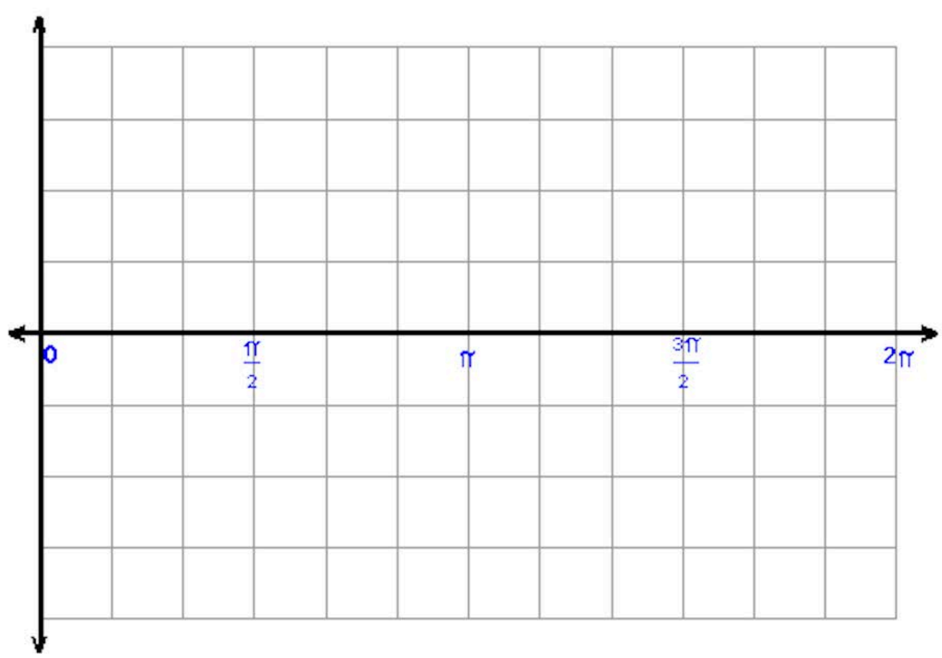
    c. Add the period of sine and cosine to your initial angle measurement (you may have to change around the fractions so that they have a common denominator before adding them).
        i. Angle measurement + period = ______________
    d. Find the terminal point of the new angle.
        i. Terminal Point: _____________
    e. Did you get the same terminal point as your initial angle? Explain why.

Now that we know the period of a basic sine and cosine function, let's figure out how we can transform it!

6. Graph one period of the function $y = \sin\theta$ below



7. Use a graphing calculator to graph the function $y = \sin 2\theta$



a. Compare the two graphs. What is different about them? What is the same about them?

b. For the graph of $y = \sin 2\theta$, how long does it take (what θ value) for the graph to complete one full cycle?

c. Therefore, what is the period of $y = \sin 2\theta$?

8. Consider the graph of the function $y = \sin \frac{1}{2}\theta$.
   a. Make a prediction about the period of this function.
   b. Graph one period of the function $y = \sin \frac{1}{2}\theta$ below

   c. Check your answers by using a graphing calculator

Period

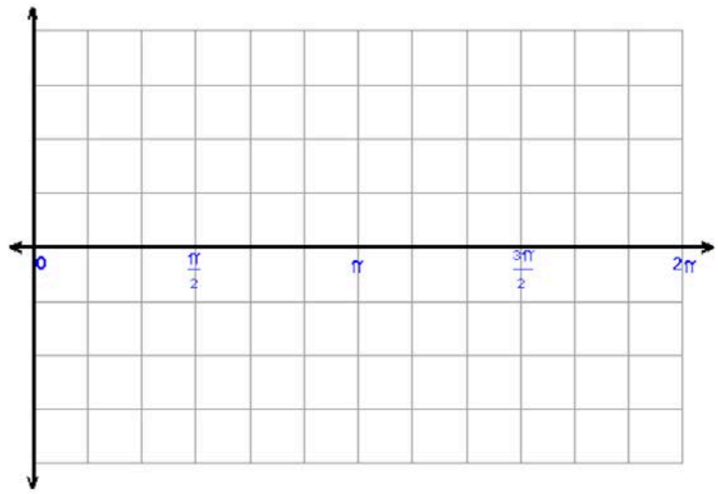
For the sine and cosine functions $f(x) = \sin(kx)$ and $g(x) = \cos(kx)$, the periods $= \frac{2\pi}{k}$.

Practice:

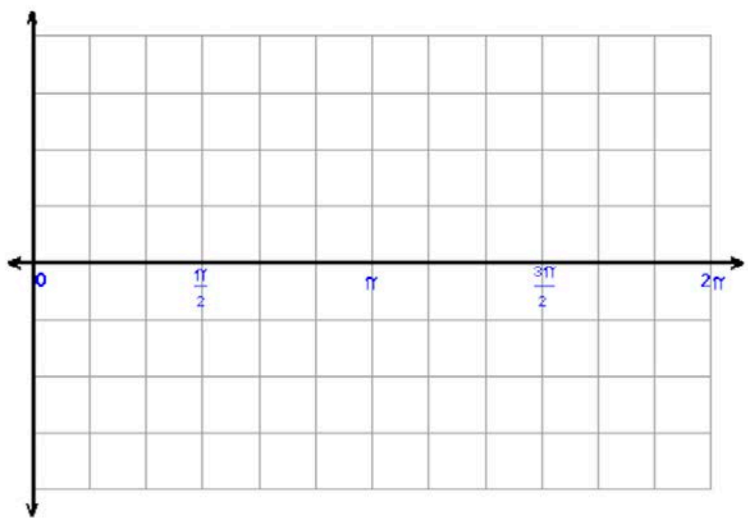
9. Find the period of the following functions
   a. $y = \cos\ .7x$
   b. $y = \sin \frac{\theta}{3}$
   c. $y = \cos 6\ \theta$
   d. $y = \cos \frac{3}{5}\theta$

10. Graph the following functions

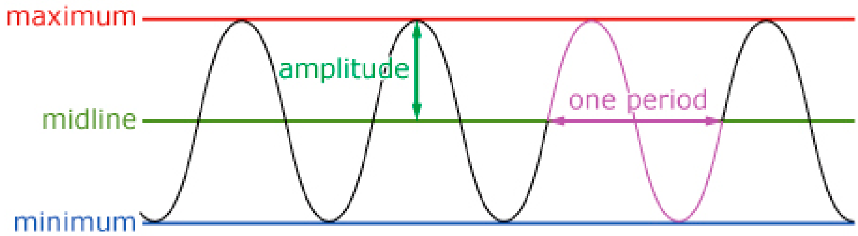
e. $y = \sin 4\theta$



f. $y = \cos 3x$

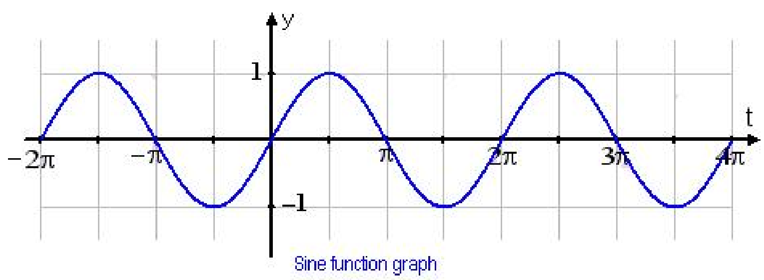


III. Amplitude



Have you ever been to a really loud concert? Volume (or loudness) plays an important role in music. Next time you listen to a song, notice how the volume of the music constantly changes. This helps to create a certain mood or feeling to the song. Of course, volume can also be represented mathematically.

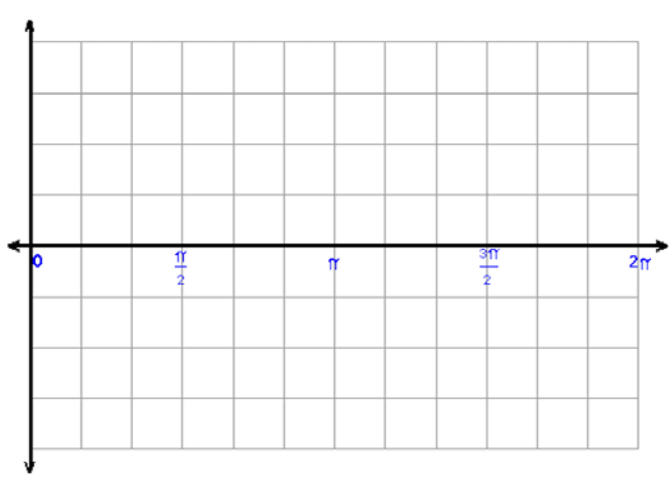
The **amplitude** of a sine or cosine function is the distance from the horizontal center (called the midline) to the highest or lowest point on the curve. Another way to think about amplitude is half the distance between the highest and lowest point.



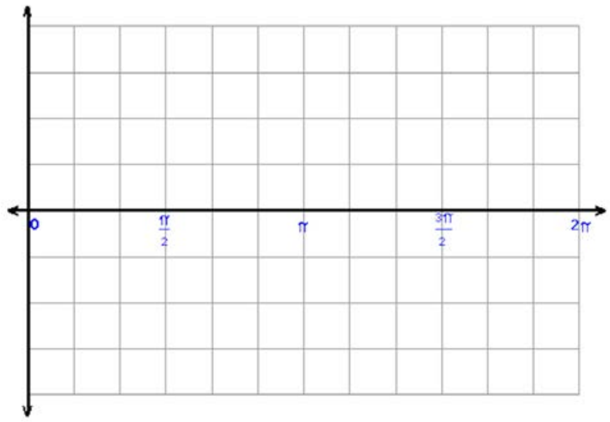
Notice in the sine graph above, the maximum y value that the curve reaches is 1, and the minimum value is -1. Therfore, the amplitude of the sine function is 1.

In your graphing calculator, graph the following functions:

11. $y = 2\cos x$



12. $y = \frac{1}{2}\sin\theta$



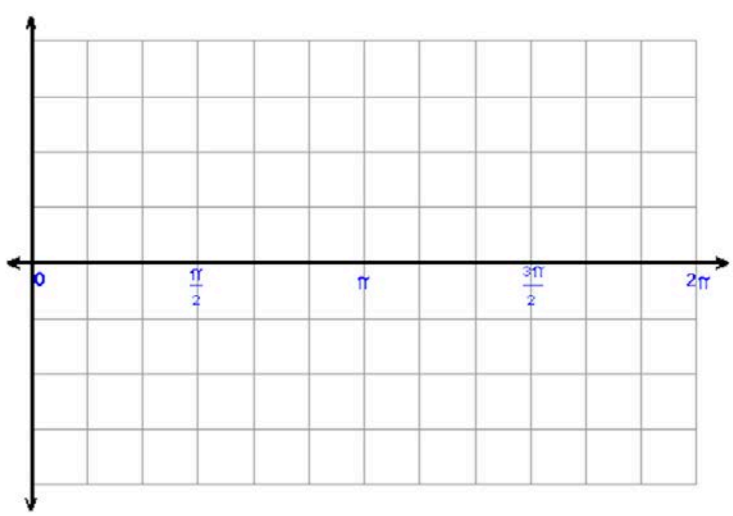
13. Make a conjecture about how to transform the amplitude of a cosine or sine function.

Amplitude

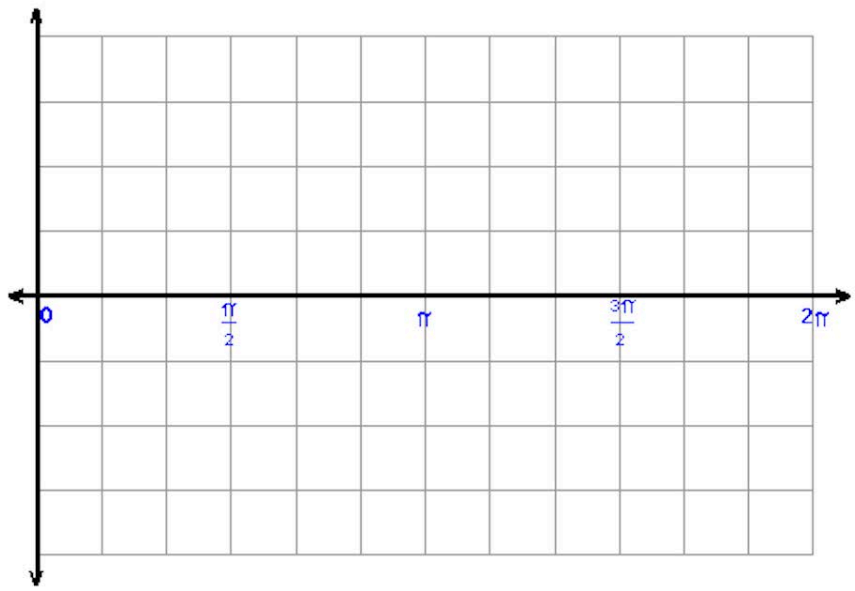
For the functions $f(x) = a \cdot \sin(x)$ and $g(x) = a \cdot \cos(kx)$, the periods $= \frac{2\pi}{k}$.

<u>Practice:</u>

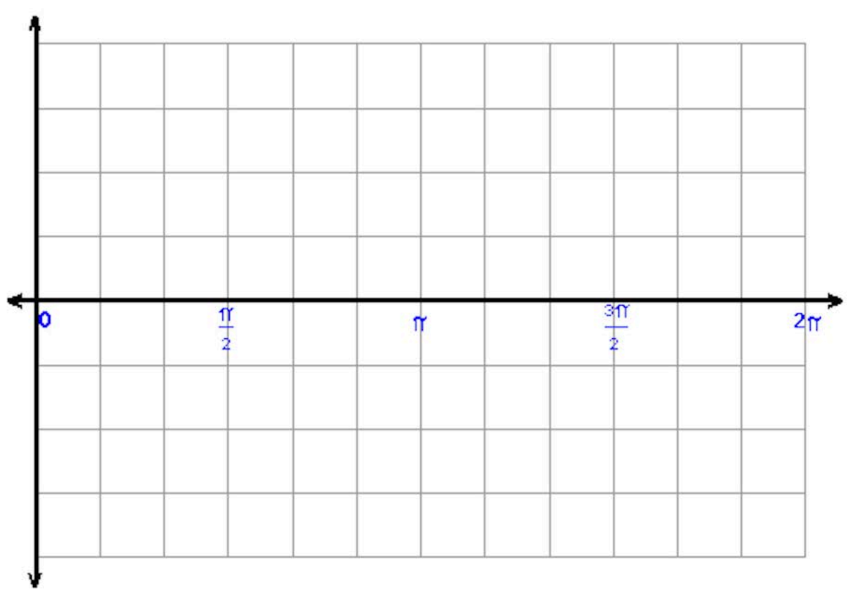
14. Graph the following functions
    a. $y = 3\cos x$



b. $y = \frac{1}{2}\sin 2x$



c. $y = 2\cos\frac{1}{2}\theta$

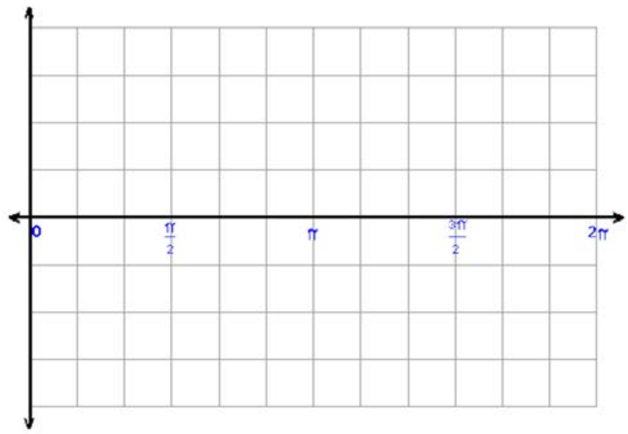


15. Graph the following functions. Then determine the period and amplitude.

a. $y = 2\cos 2x$

Amplitude:

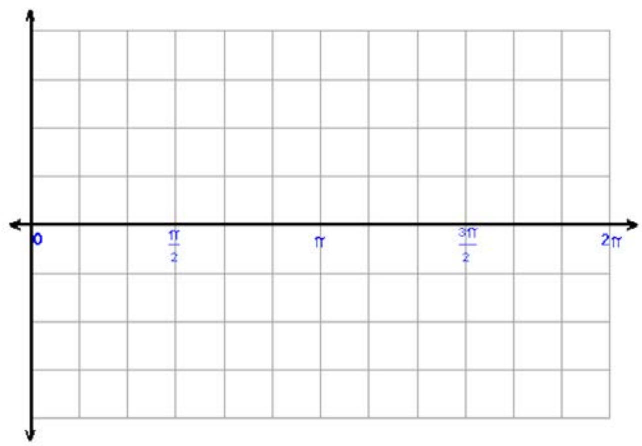
Period:



b. $f(g) = \frac{1}{2}\sin(3g)$

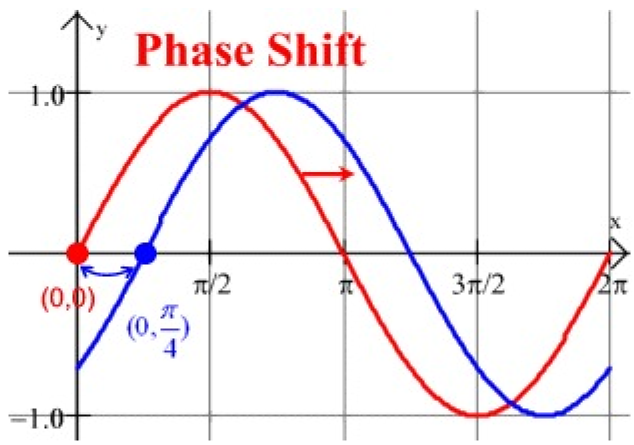
Amplitude:

Period:



IV. Horizontal and Vertical Transformations

Now that you are experts at modifying the period and amplitude of sine and cosine functions, we are going to learn about two other types of modifications: **Phase Shifts** and **Vertical Shifts**.



<u>Phase Shift</u>:

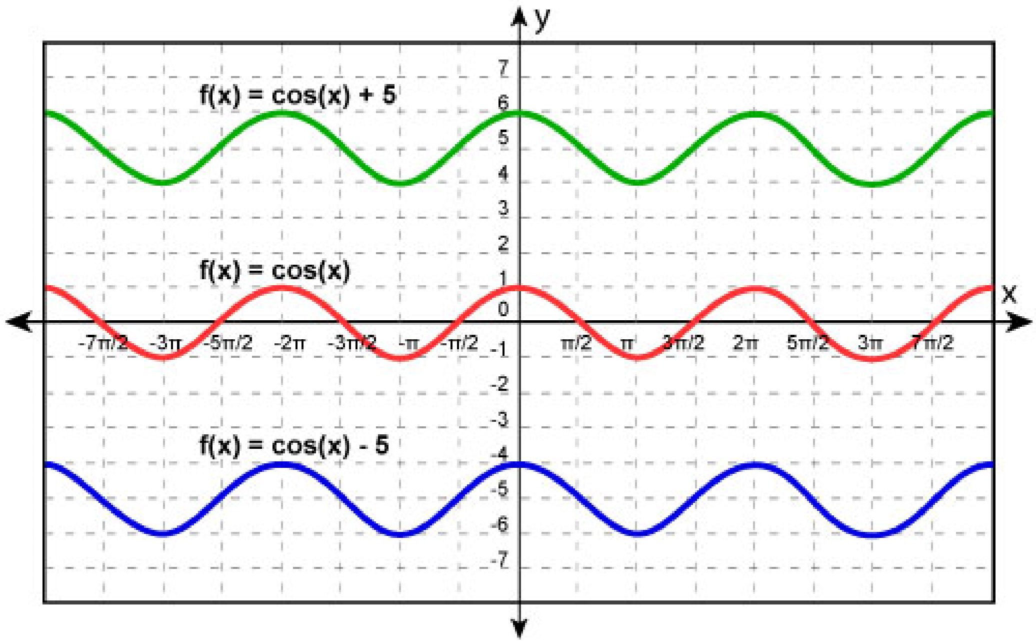
For the sine and cosine curves $y = \sin(x - b)$ and $y = \cos(x - b)$, the graph is shifted horizontally to the right if b is positive and to the left if b is negative. For example, the graph $y = 3\sin 2(x - \frac{\pi}{2})$ has an amplitude of 3, a period of π, and is shifted to the right a distance of $\frac{\pi}{2}$.

16. Describe the horizontal phase shift of the following functions

    a. $y = \cos(x + \pi)$

    b. $f(x) = \sin(x - 1)$

    c. $y = \sin(x + 2\pi)$



<u>Vertical Shift</u>:

For the sine and cosine curves $y = v + \sin(x)$ and $y = v + \cos(x)$, the graph is shifted vertically upwards if v is positive and vertically downwards if v is negative. For example, the graph $y = -2 + \cos\frac{1}{2}(x + 1)$ has an amplitude of $\frac{1}{2}$, a period of 4π, a phase shift of 1 unit left, and a vertical shift of 2 units down.

17. Describe the vertical shift of the following functions

    a. $y = 3 + \sin(x - \pi)$

    b. $y = \cos(2x) + 1$

    c. $y = -12 - \sin 3(x + 1)$

18. Describe the phase shift and the vertical shift of the following functions

    a. $y = 1 + 2\sin 2(\theta + \frac{\pi}{4})$

    b. $y = \frac{3}{2}\cos(x + \frac{\pi}{2}) - 1$