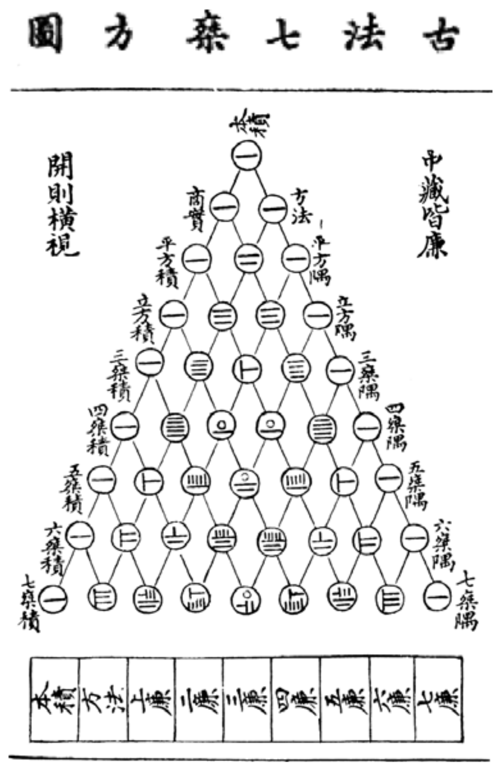
5.9: The Binomial Theorem

Pascal's Triangle



1. Show that $z = 1 + i$ is a solution to the fourth degree polynomial equation $z^4 - z^3 + 3z^2 - 4z + 6 = 0$.

2. Show that $z = 1 - i$ is a solution to the fourth degree polynomial equation $z^4 - z^3 + 3z^2 - 4z + 6 = 0$.

3. Describe any patterns you see in your work in the first two problems.

II. Binomial Expansion and Pascal's Triangle

Instead of looking at complex numbers such as $1+i$ or $1-i$, let's look at any binomial expression x + y.

4. Let's try to find a shortcut to write any expression $(a+b)$ in expanded form without having to multiply binomials repeatedly. Let's try this for a few values of n and then look for a pattern.
   a. How can we write an expression equivalent to $(a+b)^0$ in expanded form?

   b. Write $(a+b)^1$ in expanded form:

   c. Write $(a+b)^2$ in expanded form:

   d. Write $(a+b)^3$ in expanded form:

5. Describe any patterns you see.

Pascal's Triangle is a triangular configuration that represents (amongst other things) the coefficients of binomial expansion. Thus, Pascal's Triangle turns the information below…

$$(x+y)^0 = 1 \quad \text{0th row}$$
$$(x+y)^1 = 1x + 1y \quad \text{1st row}$$
$$(x+y)^2 = 1x^2 + 2xy + 1y^2 \quad \text{2nd row}$$
$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^2 \quad \text{3rd row}$$
$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 \quad \text{4th row}$$
$$(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5 \quad \text{5th row}$$

…into this simple model:

Row 0:						1						
Row 1:					1		1					
Row 2:				1		2		1				
Row 3:			1		3		3		1			
Row 4:		1		4		6		4		1		
Row 5:	1		5		10		10		5		1	
Row 6:												
Row 7:												
Row 8:												

6. Complete the next three rows of Pascal's Triangle. This can be done recursively from the row above. If you don't see the pattern, ask a neighbor. If you still don't get it look at the diagram at the end of this document.

7. What <u>other</u> patterns do you see in Pascal's Triangle?

II. Combinatorics and Factorials

Pascal's Triangle can show you how many combinations of objects are possible.

EXAMPLE 2: You have 16 pool balls. How many different ways could you choose just 3 of them (ignoring the order that you select them)?

Answer: go down to the start of row 16 (the top row is 0), and then along 3 places (the first place is 0) and the value there is your answer, **560**.

Here is an extract of rows 14-16 of Pascal's Triangle:

```
        1    14    91    364  ...
     1    15    105   455   1365  ...
   1    16   120   560   1820  4368  ...
```

8. How many different ways can you pick 4 prizes out of a grab bag of 15? 1365

There is a way to calculate an entry of Pascal's triangle without writing out the whole triangle, but we first need the idea of a *factorial*, which we denote by $n!$ for integers $n \geq 0$. First, we define $0!=1$. Then, if $n>0$, we define $n!$ to be the product of all positive integers less than or equal to n.

EXAMPLE 3: $2!=2 \cdot 1 = 2$ and $3!=3 \cdot 2 \cdot 1=6$.

Your Turn:

9. Calculate the following factorials.
   a. 6!

   b. 10!

10. Calculate the value of the following factorial expressions.

a. $\frac{7!}{6!}$

b. $\frac{10!}{6!}$

c. $\frac{8!}{5!}$

d. $\frac{12!}{10!}$

To calculate the number of possible combinations C of n objects selected in k groups we use the formula $C(n,k) = \frac{n!}{k!(n-k)!}$, where $n \geq 0$ and $k \geq n$

EXAMPLE 4: How many ways can you select 4 objects from a group of 6?

$$C(6,4) = \frac{6!}{4!\,(6-4)!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(2 \cdot 1)} = \frac{6 \cdot 5}{2 \cdot 1} = 15$$

11. Calculate the following quantities.

a. $C(1,0)$ and $C(1,1)$

b. $C(2,0)$, $C(2,1)$, and $C(2,2)$

c. $C(3,0)$, $C(3,1)$, $C(3,2)$, and $C(3,3)$

d. $C(4,0)$, $C(4,1)$, $C(4,2)$, $C(4,3)$, and $C(4,4)$

12. What patterns do you see in Question 11?

13. Expand the expression $(u + v)^3$.

14. Expand the expression $(u + v)^4$.

15.

a. Multiply the expression you wrote in Exercise 13 by u.

b. Multiply the expression you wrote in Exercise 14 by v.

c. How can you use the results from parts (a) and (b) to find the expanded form of the expression $(u + v)^5$?

16. What do you notice about your expansions for $(u+v)^4$ and $(u+v)^5$? Does your observation hold for other powers of $(u+v)$?

17.

a. $(x+y)^6$

b. $(x+2y)^3$

c. $(ab+bc)^4$

d. $(3xy-2z)^3$

e. $(4p^2qr-qr^2)^5$

III. Return of the Triangle

18. Write the first six rows of Pascal's triangle. Then, use the triangle to find the coefficients of the terms with the powers of u and v shown, assuming that all expansions are in the form $(u + v)^n$. Explain how Pascal's triangle allows you to determine the coefficient.

a. u^2v^4

b. u^3v^2

c. u^2v^2

d. v^{10}

19. Look at the alternating sums of the rows of Pascal's triangle. An alternating sum alternately subtracts and then adds values. For example, the alternating sum of Row 2 would be $1 - 2 + 1$, and the alternating sum of Row 3 would be $1 - 3 + 3 - 1$.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

a. Compute the alternating sum for each row of the triangle shown.

b. Use the binomial theorem to explain why each alternating sum of a row in Pascal's triangle is 0.

20. Consider the Rows 0–6 of Pascal's triangle.

a. Find the sum of each row.

SUM

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

b. What pattern do you notice in the sums computed?

c. Use the binomial theorem to explain this pattern.

21. Consider the expression 11^n.

a. Calculate 11^n, where $n = 0, 1, 2, 3, 4$.

b. What pattern do you notice in the successive powers?

c. Use the binomial theorem to demonstrate why this pattern arises.

d. Use a calculator to find the value of 11^5. Explain whether this value represents what would be expected based on the pattern seen in lower powers of 11.

Lesson Summary

Pascal's triangle is an arrangement of numbers generated recursively:

Row 0:						1					
Row 1:					1		1				
Row 2:				1		2		1			
Row 3:			1		3		3		1		
Row 4:		1		4		6		4		1	
Row 5:	1		5		10		10		5		1
	⋮		⋮		⋮		⋮		⋮		⋮

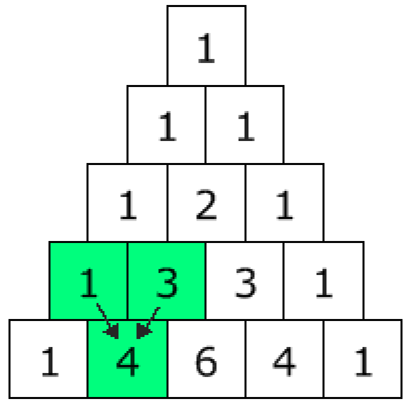
For an integer $n \geq 1$, the number $n!$ is the product of all positive integers less than or equal to n.
We define $0! = 1$.

The binomial coefficients $C(n, k)$ are given by $C(n, k) = \frac{n!}{k!(n-k)}!$ for integers $n \geq 0$ and $0 \leq k \leq n$.

THE BINOMIAL THEOREM: For any expressions u and v,

$(u + v)^n = u^n + C(n, 1)u^{n-1}v + C(n, 2)u^{n-2}v^2 + \cdots + C(n, k)u^{n-k}v^k + \cdots + C(n, n - 1)u\, v^{n-1} + v^n.$

That is, the coefficients of the expanded binomial $(u + v)^n$ are exactly the numbers in Row n of Pascal's triangle.


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1