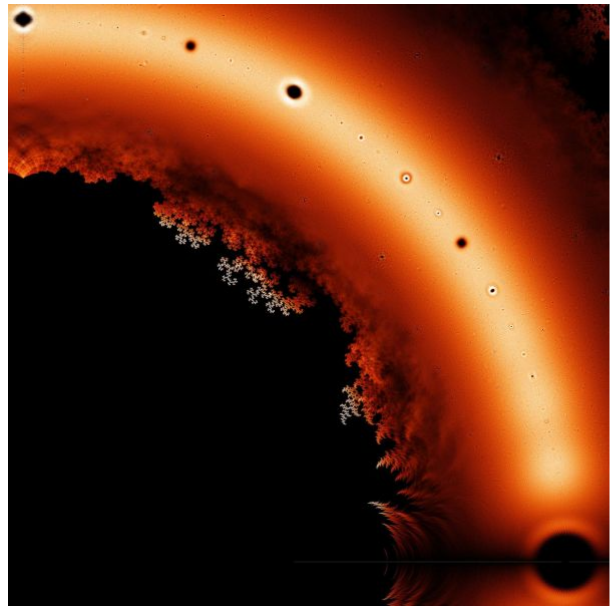
5.7: Polynomial Roots, Revised

Complex Roots



In Algebra II, you studied polynomial equations and the nature of the solutions of these equations. In this lesson you will discover what it means to solve polynomial equations over the set of complex numbers.

The image on the right is a close-up of a graph of a large number polynomial equations, with some interesting fractal qualities. See the original images from Dan Christensen on his University of Western Ontario web site.

You can also check out a slide show at: http://www.scientificamerican.com/article/math-polynomial-roots/

Solutions to Polynomial Equations

1. How many solutions are there to the equation $x^2 = 1$? Explain how you know.

2. Prove that 1 and -1 are the only solutions to the equation $x^2 = 1$.
   Let $x = a + bi$ be a complex number so that $x^2 = 1$.

   a. Substitute $a + bi$ for x in the equation $x^2 = 1$.

   b. Rewrite both sides in standard form for a complex number.

c. Equate the real parts on each side of the equation and equate the imaginary parts on each side of the equation.

d. Solve for a and b, and find the solutions for $x = a + bi$.

3. Find the product.

   a. $(z-2)(z+2)$

   b. $(z+3i)(z-3i)$

4. Write each of the following quadratic expressions as the product of two linear factors.

   a. $z^2 - 4$

   b. $z^2 + 4$

   c. $z^2 - 4i$

   d. $z^2 + 4i$

5. Can a quadratic polynomial equation with real coefficients have one real solution and one complex solution? If so, give an example of such an equation. If not, explain why not.

Recall from Algebra II that every quadratic expression can be written as a product of two linear factors, that is,

$$ax^2 + bx + c = a(x - r_1)(x - r_2),$$

where r_1 and r_2 are solutions of the polynomial equation $ax^2 + bx + c = 0$.

6. Solve each equation by factoring, and state the solutions.
   a. $x^2 + 25 = 0$

   b. $x^2 + 10x + 25 = 0$

7. Give an example of a quadratic equation with $2 + 3i$ as one of its solutions.

8. A quadratic polynomial equation with real coefficients has a complex solution of the form $a + bi$ with $b \neq 0$. What must its other solution be, and why?

9. Write the left side of each equation as a product of linear factors, and state the solutions.
   a. $x^3 - 1 = 0$

   b. $x^3 + 8 = 0$

   c. $x^4 + 7x^2 + 10 = 0$

10. Consider the polynomial $p(x) = x^3 + 4x^2 + 6x - 36$.
   a. Graph $y = x^3 + 4x^2 + 6x - 36$, and find the real zero of polynomial p.

   b. Write $p(x)$ as a product of linear factors.

   c. What are the solutions to the equation $p(x) = 0$?

11. Malaya was told that the volume of a box that is a cube is 4,096 cubic inches. She knows the formula for the volume of a cube with side length x is $V(x) = x^3$, so she models the volume of the box with the equation $x^3 - 4096 = 0$.

    a. Solve this equation for x.

    b. Malaya shows her work to Tiffany and tells her that she has found three different values for the side length of the box. Tiffany looks over Malaya's work and sees that it is correct but explains to her that there is only one valid answer. Help Tiffany explain which answer is valid and why.

12. Consider the polynomial $p(x) = x^6 - 2x^5 + 7x^4 - 10x^3 + 14x^2 - 8x + 8$.

    a. Graph $y = x^6 - 2x^5 + 7x^4 - 10x^3 + 14x^2 - 8x + 8$, and state the number of real zeros of p.

    b. Verify that i is a zero of p.

c. Given that i is a zero of p, state another zero of p.

d. Given that $2i$ and $1 + i$ are also zeros of p, explain why polynomial p cannot possibly have any real zeros.

e. What is the solution set to the equation $p(x) = 0$?

13. Think of an example of a sixth degree polynomial equation that when written in standard form has integer coefficients, four real number solutions, and two imaginary number solutions. How can you be sure your equation will have integer coefficients?

Lesson Summary

Relevant Vocabulary

POLYNOMIAL FUNCTION: Given a polynomial expression in one variable, a *polynomial function in one variable* is a function $f: \mathbb{R} \to \mathbb{R}$ such that for each real number x in the domain, $f(x)$ is the value found by substituting the number x into all instances of the variable symbol in the polynomial expression and evaluating.

It can be shown that if a function $f: \mathbb{R} \to \mathbb{R}$ is a polynomial function, then there is some nonnegative integer n and collection of real numbers $a_0, a_1, a_2, \dots, a_n$ with $a_n \neq 0$ such that the function satisfies the equation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

for every real number x in the domain, which is called the *standard form of the polynomial function.* The function $f(x) = 3x^3 + 4x^2 + 4x + 7$, where x can be any real number, is an example of a function written in standard form.

DEGREE OF A POLYNOMIAL FUNCTION: The *degree of a polynomial function* is the degree of the polynomial expression used to define the polynomial function. The degree is the highest degree of its terms.

The degree of $f(x) = 8x^3 + 4x^2 + 7x + 6$ is 3, but the degree of $g(x) = (x+1)^2 - (x-1)^2$ is 1 because when g is put into standard form, it is $g(x) = 4x$.

ZEROS OR ROOTS OF A FUNCTION: A *zero* (or *root*) of a function $f: \mathbb{R} \to \mathbb{R}$ is a number x of the domain such that $f(x) = 0$. A zero of a function is an element in the solution set of the equation $f(x) = 0$.

Given any two polynomial functions p and q, the solution set of the equation $p(x)q(x) = 0$ can be quickly found by solving the two equations $p(x) = 0$ and $q(x) = 0$ and combining the solutions into one set.

A number a is zero of a polynomial function p with multiplicity m if the factored form of p contains $(x - a)^m$.

Every polynomial function of degree n, for $n \geq 1$, has n zeros over the complex numbers, counted with multiplicity. Therefore, such polynomials can always be factored into n linear factors.