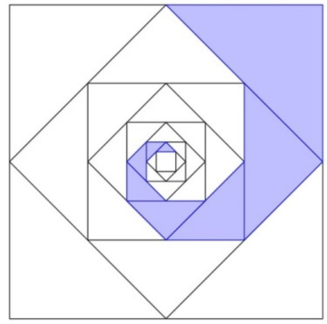
5.6 Even More Complex Multiplication

Practice Tasks



I. Concepts and Procedures

1. Find the modulus and argument for each of the following complex numbers.

   a. $z_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

   b. $z_2 = 2 + 2\sqrt{3}i$

   c. $z_3 = -3 + 5i$

   d. $z_4 = -2 - 2i$

   e. $z_5 = 4 - 4i$

   f. $z_6 = 3 - 6i$

2. For parts (a)–(c), determine the geometric effect of the specified transformation.

   a. $L(z) = -3z$

   b. $L(z) = -100z$

   c. $L(z) = -\frac{1}{3}z$

   d. Describe the geometric effect of the transformation $L(z) = az$ for any negative real number a.

3. For parts (a)–(c), determine the geometric effect of the specified transformation.

a. $L(z) = (-3i)z$

b. $L(z) = (-100i)z$

c. $L(z) = \left(-\frac{1}{3}i\right)z$

d. Describe the geometric effect of the transformation $L(z) = (bi)z$ for any negative real number b.

4. A function L is a **linear transformation** if all z and w in the domain of L and all constants a meet the following two conditions:
   i. $L(z + w) = L(z) + L(w)$
   ii. $L(az) = aL(z)$

   Show that the following functions meet the definition of a linear transformation.

   a. $L_1(z) = 4z$

   b. $L_2(z) = iz$

   c. $L_3(z) = (4 + i)z$

5. Suppose that we have two linear transformations $L_1(z) = 3z$ and $L_2(z) = (5i)z$.
   a. What is the geometric effect of first performing transformation L_1, and then performing transformation L_2?

   b. What is the geometric effect of first performing transformation L_2, and then performing transformation L_1?

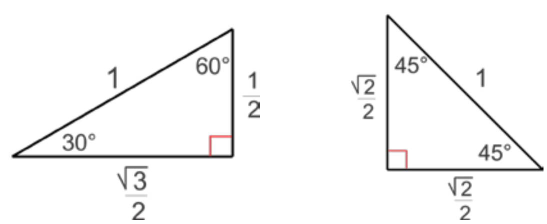
   c. Are your answers to parts (a) and (b) the same or different? Explain how you know.

6. Suppose that we have two linear transformations $L_1(z) = (4 + 3i)z$ and $L_2(z) = -z$.

What is the geometric effect of first performing transformation L_1, and then performing transformation L_2?

7. Suppose that we have two linear transformations $L_1(z) = (3 - 4i)z$ and $L_2(z) = -z$. What is the geometric effect of first performing transformation L_1, and then performing transformation L_2?

8. Explain the geometric effect of the linear transformation $L(z) = (a - bi)z$, where a and b are positive real numbers.



9. In Geometry, we learned the special angles of a right triangle whose hypotenuse is 1 unit. The figures are shown above. Describe the geometric effect of the following transformations.

   a. $L_1(z) = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) z$

   b. $L_2(z) = \left(2 + 2\sqrt{3}i\right)z$

   c. $L_3(z) = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) z$

   d. $L_4(z) = (4 + 4i)z$

10. The vertices $A(0,0)$, $B(1,0)$, $C(1,1)$, $D(0,1)$ of a unit square can be represented by the complex numbers $A = 0, B = 1, C = 1 + i, D = i$. We learned that multiplication of those complex numbers by i rotates the unit square by 90° counterclockwise. What do you need to multiply by so that the unit square will be rotated by 90° clockwise?

11. Find a linear transformation L that will have the geometric effect of rotation by the specified amount without dilating.

    a. 45° counterclockwise

    b. 60° counterclockwise

    c. 180° counterclockwise

    d. 120° counterclockwise

    e. 30° clockwise

    f. 90° clockwise

    g. 180° clockwise

    h. 135° clockwise

12. Suppose that we have linear transformations L_1 and L_2 as specified below. Find a formula for $L_2(L_1(z))$ for complex numbers z.

    a. $L_1(z) = (1 + i)z$ and $L_2(z) = (1 - i)z$

    b. $L_1(z) = (3 - 2i)z$ and $L_2(z) = (2 + 3i)z$

    c. $L_1(z) = (-4 + 3i)z$ and $L_2(z) = (-3 - i)z$

    d. $L_1(z) = (a + bi)z$ and $L_2(z) = (c + di)z$ for real numbers a, b, c and d.

II. Problem Solving, Reasoning and Modeling

1. In the lesson, you justified your observation that the geometric effect of a transformation $L(z) = wz$ is a rotation by $\arg(w)$ and a dilation by $|w|$ for a complex number w that is represented by a point in the first quadrant of the coordinate plane. In this exercise, you will verify that this observation is valid for any complex number w. For a complex number $w = a + bi$, we only considered the case where $a > 0$ and $b > 0$. There are eight additional possibilities we need to consider.

   a. Case 1: The point representing w is the origin. That is, $a = 0$ and $b = 0$.

   In this case, explain why $L(z) = (a + bi)z$ has the geometric effect of rotation by $\arg(a + bi)$ and dilation by $|a + bi|$.

   b. Case 2: The point representing w lies on the positive real axis. That is, $a > 0$ and $b = 0$.
   In this case, explain why $L(z) = (a + bi)z$ has the geometric effect of rotation by $\arg(a + bi)$ and dilation by $|a + bi|$.

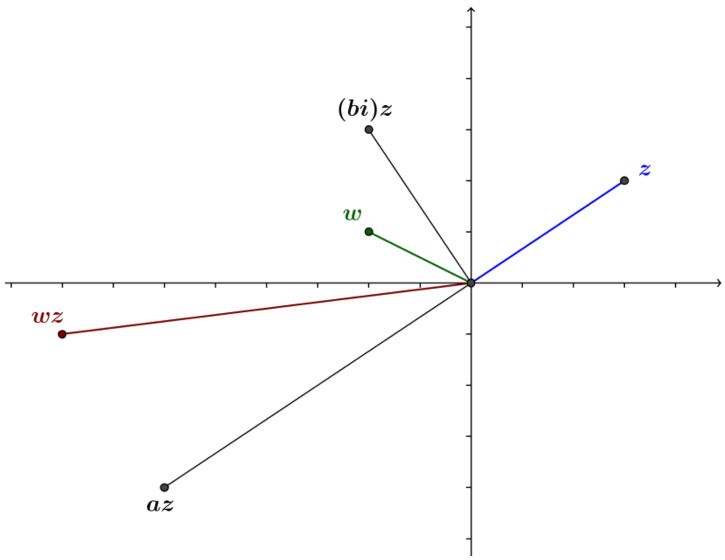
   c. Case 3: The point representing w lies on the negative real axis. That is, $a < 0$ and $b = 0$.
   In this case, explain why $L(z) = (a + bi)z$ has the geometric effect of rotation by $\arg(a + bi)$ and dilation by $|a + bi|$.

   d. Case 4: The point representing w lies on the positive imaginary axis. That is, $a = 0$ and $b > 0$.

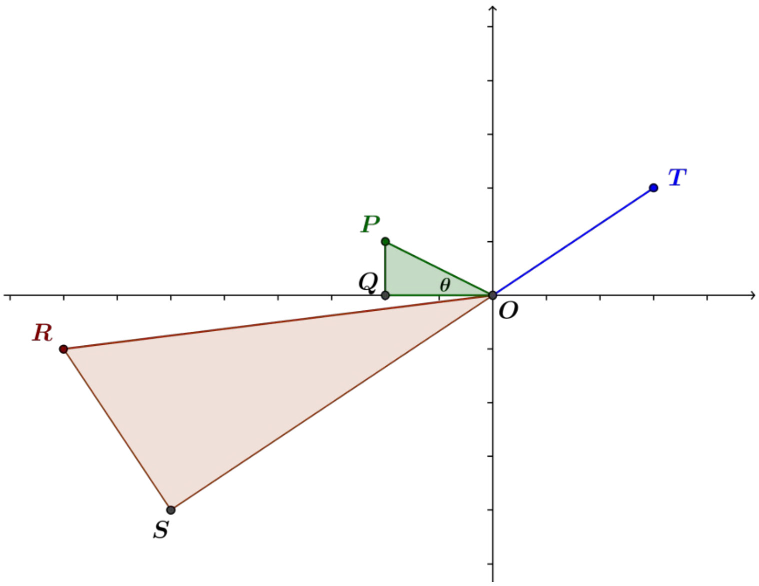
   In this case, explain why $L(z) = (a + bi)z$ has the geometric effect of rotation by $\arg(a + bi)$ and dilation by $|a + bi|$.

e. Case 5: The point representing w lies on the negative imaginary axis. That is, $a = 0$ and $b < 0$.
In this case, explain why $L(z) = (a + bi)z$ has the geometric effect of rotation by $\arg(a + bi)$ and dilation by $|a + bi|$.

f. Case 6: The point representing $w = a + bi$ lies in the second quadrant. That is, $a < 0$ and $b > 0$. Points representing , z, az, $(bi)z$, and $wz = az + (bi)z$ are shown in the figure below.



For convenience, rename the origin O and let $P = w$, $Q = a$, $R = wz$, $S = az$, and $T = z$, as shown below. Let $m(\angle POQ) = \theta$.



i. Argue that $\triangle OPQ \sim \triangle ORS$.

ii. Express the argument of az in terms of $\arg(z)$.

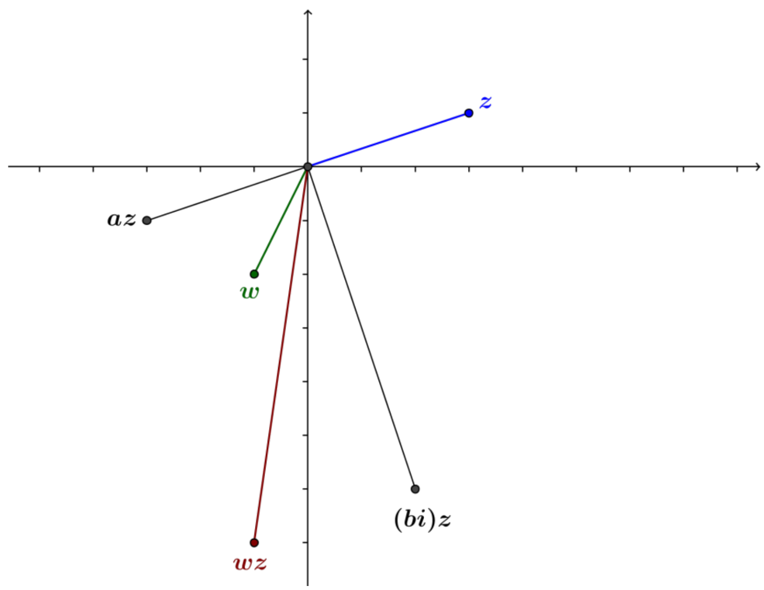
iii. Express $\arg(w)$ in terms of θ, where $\theta = m(\angle POQ)$.

iv. Explain why $\arg(wz) = \arg(az) - \theta$.

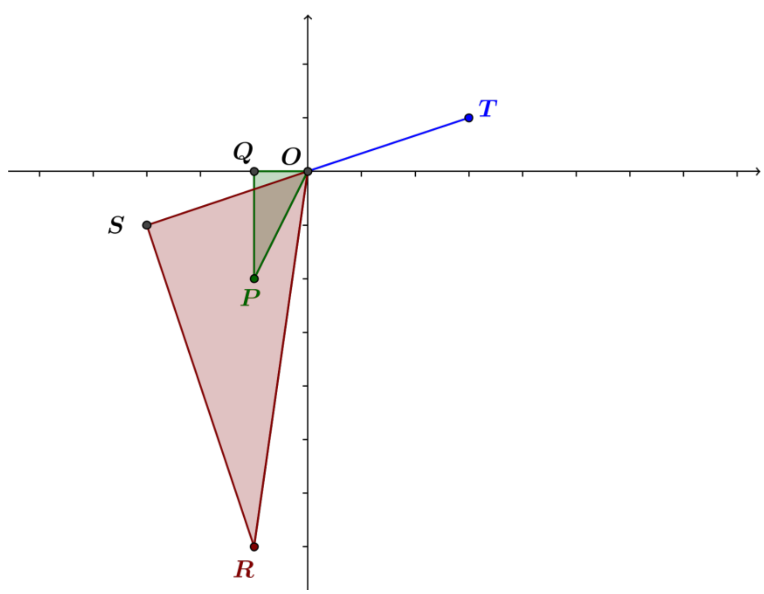
v. Combine your responses from parts (ii), (iii) and (iv) to express $\arg(wz)$ in terms of $\arg(z)$ and $\arg(w)$.

g. Case 7: The point representing $w = a + bi$ lies in the third quadrant. That is $a < 0$ and $b < 0$.

Points representing , z, az, $(bi)z$, and $wz = az + (bi)z$ are shown in the figure below.



For convenience, rename the origin O and let $P = w$, $Q = a$, $R = wz$, $S = az$, and $T = z$, as shown below. Let $m(\angle POQ) = \theta$.



i. Argue that $\triangle OPQ \sim \triangle ORS$.

ii. Express the argument of az in terms of $\arg(z)$.

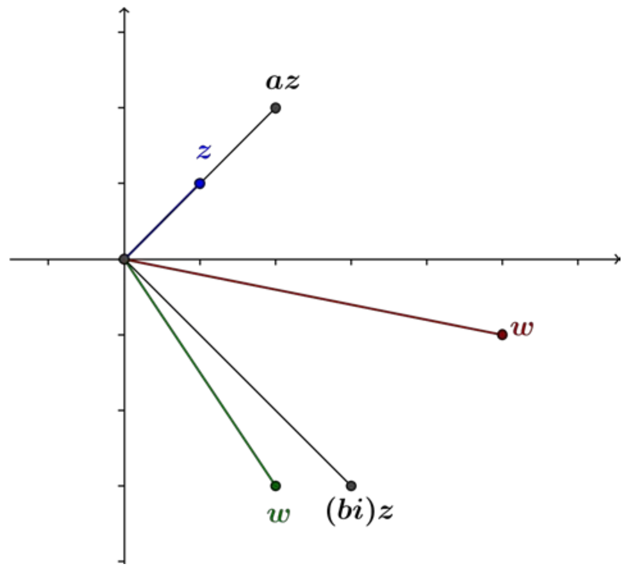
iii. Express $\arg(w)$ in terms of θ, where $\theta = m(\angle POQ)$.

iv. Explain why $\arg(wz) = \arg(az) + \theta$.

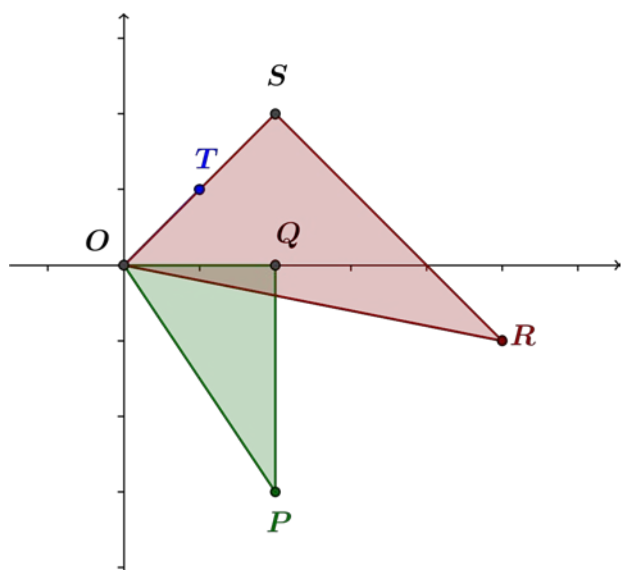
v. Combine your responses from parts (ii), (iii), and (iv) to express $\arg(wz)$ in terms of $\arg(z)$ and $\arg(w)$.

h. Case 8: The point representing $w = a + bi$ lies in the fourth quadrant. That is, $a > 0$ and $b < 0$.

Points representing , z, az, $(bi)z$, and $wz = az + (bi)z$ are shown in the figure below.



For convenience, rename the origin O, and let $P = w$, $Q = a$, $R = wz$, $S = az$, and $T = z$, as shown below. Let $m(\angle POQ) = \theta$.



i. Argue that $\triangle\ OPQ \sim \triangle\ ORS$.

ii. Express $\arg(w)$ in terms of θ, where $\theta = m(\angle POQ)$.

iii. Explain why $m(\angle QOR) = \theta - \arg(z)$.

iv. Express $\arg(wz)$ in terms of $m(\angle QOR)$

v. Combine your responses from parts (ii), (iii), and (iv) to express $\arg(wz)$ in terms of $\arg(z)$ and $\arg(w)$.

2. Summarize the results of Problem 1, parts (a)–(h) and the lesson.