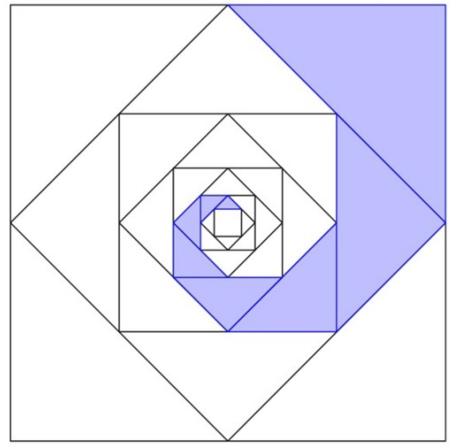


5.6: Even More Complex Multiplication

Discovering the Geometric Effects

The design on the right – without the colored shading – is often found in architectural decoration. It is known as the *ad quadratum*, where one square is set diagonally inside another square.

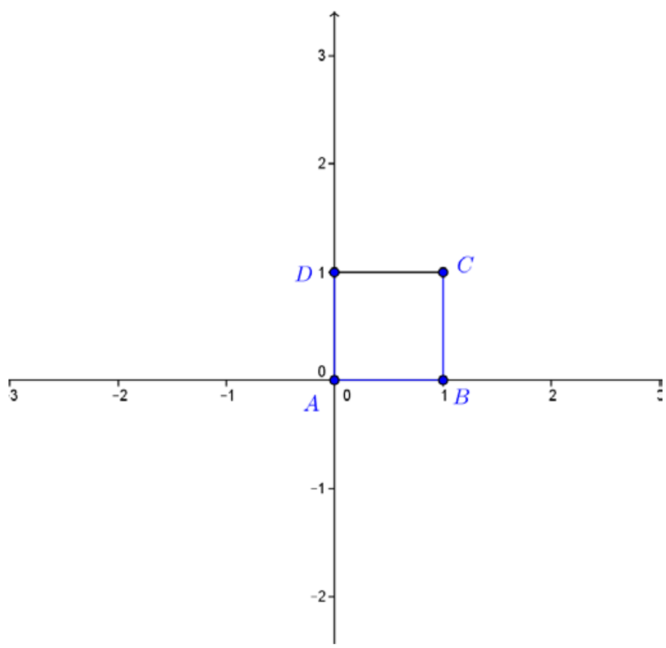
1. How many squares do you see?
2. If the length of a side of the largest square is 2, find the length of the side of each smaller square.
   Write your answer as a series:

II. Unit Square Transformations

The vertices $A(0,0)$, $B(1,0)$, $C(1,1)$, and $D(0,1)$ of a unit square can be represented by the complex numbers $A = 0$, $B = 1$, $C = 1 + i$, and $D = i$.

3. Let $L_1(z) = -z$.

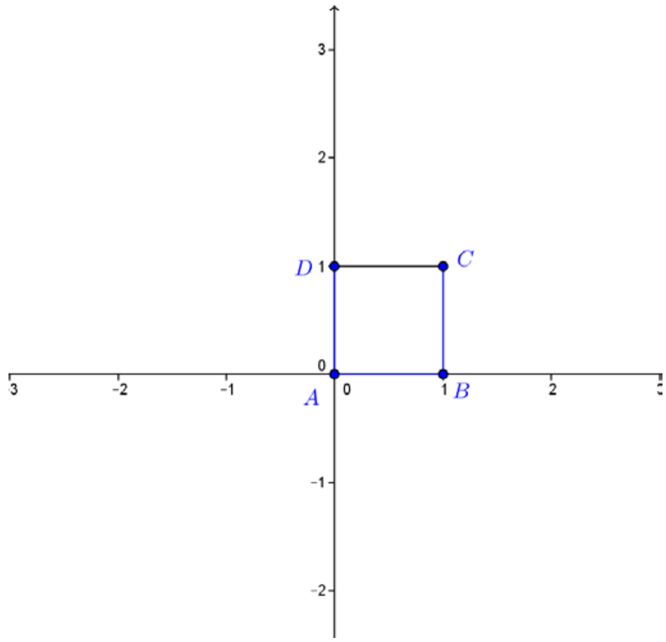
   a. Calculate $A' = L_1(A)$, $B' = L_1(B)$, $C' = L_1(C)$, and $D' = L_1(D)$. Plot these four points on the axes.



   b. Describe the geometric effect of the transformation $L_1(z) = -z$ on the square $ABCD$.

4. Let $L_2(z) = 2z$.

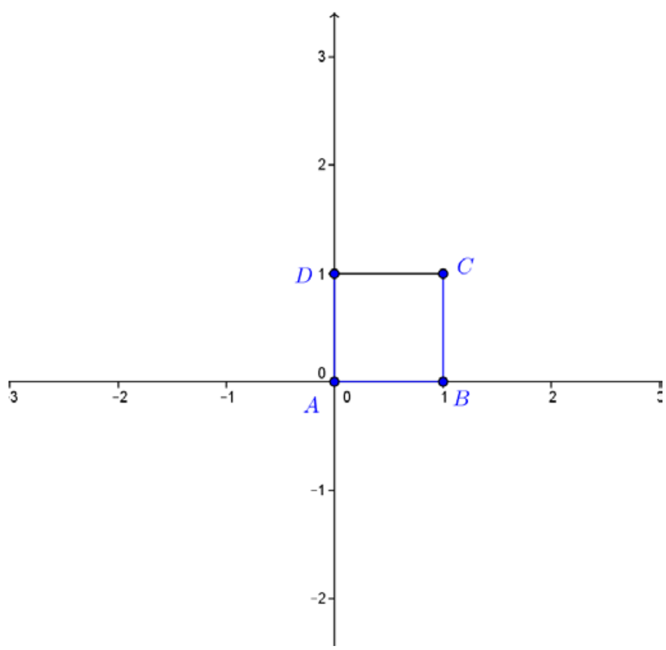
a. Calculate $A' = L_2(A)$, $B' = L_2(B)$, $C' = L_2(C)$, and $D' = L_2(D)$. Plot these four points on the axes.



b. Describe the geometric effect of the transformation $L_2(z) = 2z$ on the square $ABCD$.

5. Let $L_3(z) = iz$.

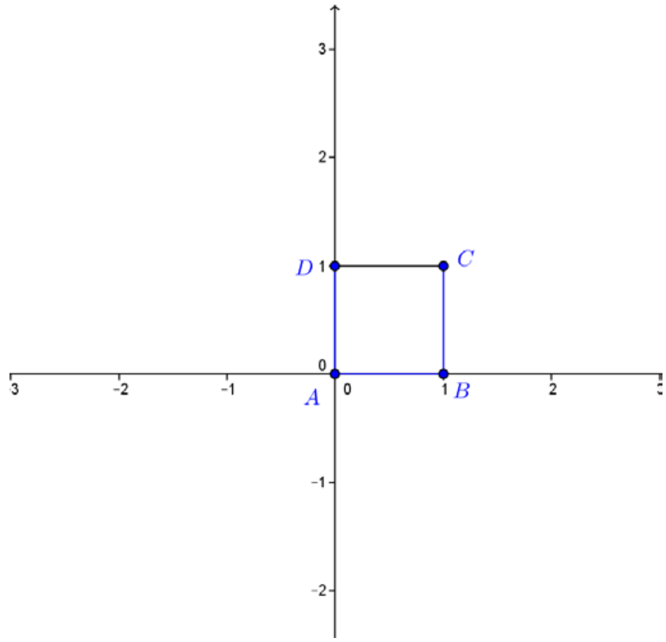
a. Calculate $A' = L_3(A)$, $B' = L_3(B)$, $C' = L_3(C)$, and $D' = L_3(D)$. Plot these four points on the axes.



b. Describe the geometric effect of the transformation $L_3(z) = iz$ on the square $ABCD$.

6. Let $L_4(z) = (2i)z$.
   a. Calculate $A' = L_4(A)$, $B' = L_4(B)$, $C' = L_4(C)$, and $D' = L_4(D)$. Plot these four points on the axes.

   b. Describe the geometric effect of the transformation $L_4(z) = (2i)z$ on the square $ABCD$.



7. Explain how transformations L_2, L_3, and L_4 are related.

8. We will continue to use the unit square $ABCD$ with $A = 0, B = 1, C = 1 + I, D = i$ for this exercise.
   a. What is the geometric effect of the transformation $L(z) = 5z$ on the unit square?

   b. What is the geometric effect of the transformation $L(z) = (5i)z$ on the unit square?

   c. What is the geometric effect of the transformation $L(z) = (5i^3)z$ on the unit square?

   d. What is the geometric effect of the transformation $L(z) = (5i^4)z$ on the unit square?

   e. What is the geometric effect of the transformation $L(z) = (5i^5)z$ on the unit square?

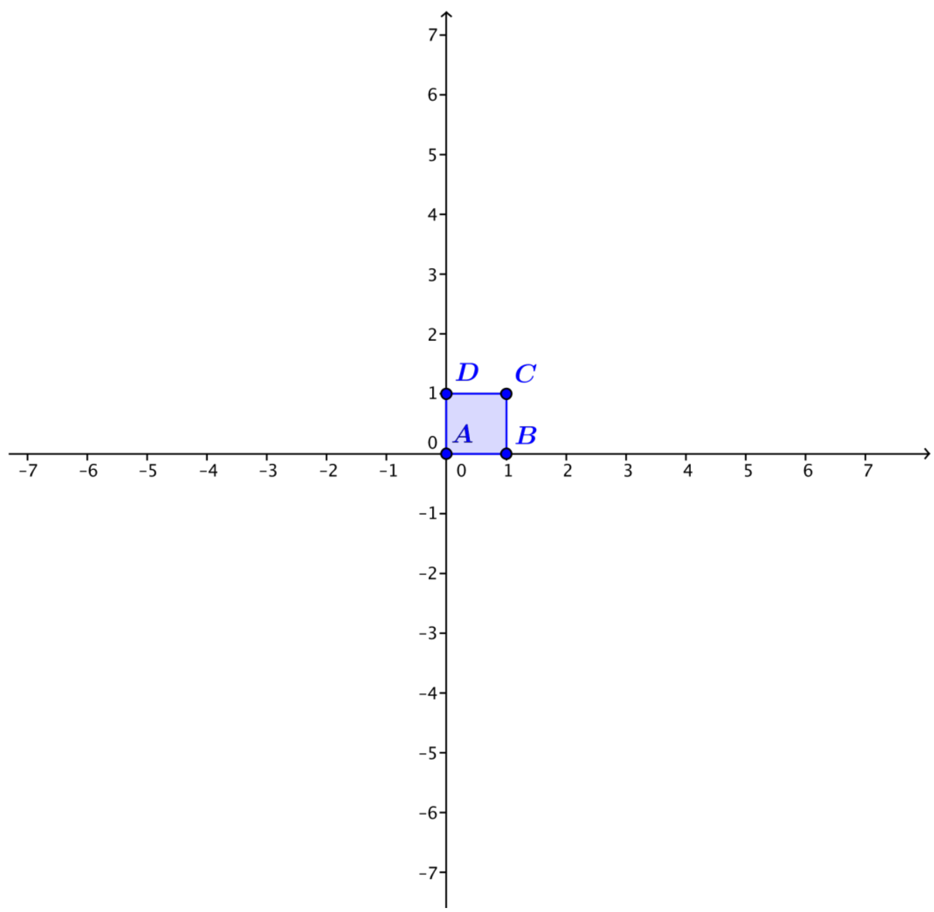
   f. What is the geometric effect of the transformation $L(z) = (5i^n)z$ on the unit square, for some integer $n \geq 0$?

III. Exploratory Challenge

Your group is assigned either to the 1-team, 2-team, 3-team, or 4-team. Each team will answer the questions below for the transformation that corresponds to their team number:

$$L_1(z) = (3 + 4i)z$$
$$L_2(z) = (-3 + 4i)z$$
$$L_3(z) = (-3 - 4i)z$$
$$L_4(z) = (3 - 4i)z.$$

9. The unit square unit square $ABCD$ with $A = 0, B = 1, C = 1 + i, D = i$ is shown below. Apply your transformation to the vertices of the square $ABCD$ and plot the transformed points A', B', C', and D' on the same coordinate axes.



For the 1-team:	For the 2-team:
a. Why is $B' = 3 + 4i$? b. What is the argument of $3 + 4i$? c. What is the modulus of $3 + 4i$?	a. Why is $B' = -3 + 4i$? b. What is the argument of $-3 + 4i$? c. What is the modulus of $-3 + 4i$?
For the 3-team: a. Why is $B' = -3 - 4i$? b. What is the argument of $-3 - 4i$? c. What is the modulus of $-3 - 4i$?	For the 4-team: a. Why is $B' = 3 - 4i$? b. What is the argument of $3 - 4i$? c. What is the modulus of $3 - 4i$?

10. All groups should also answer the following:

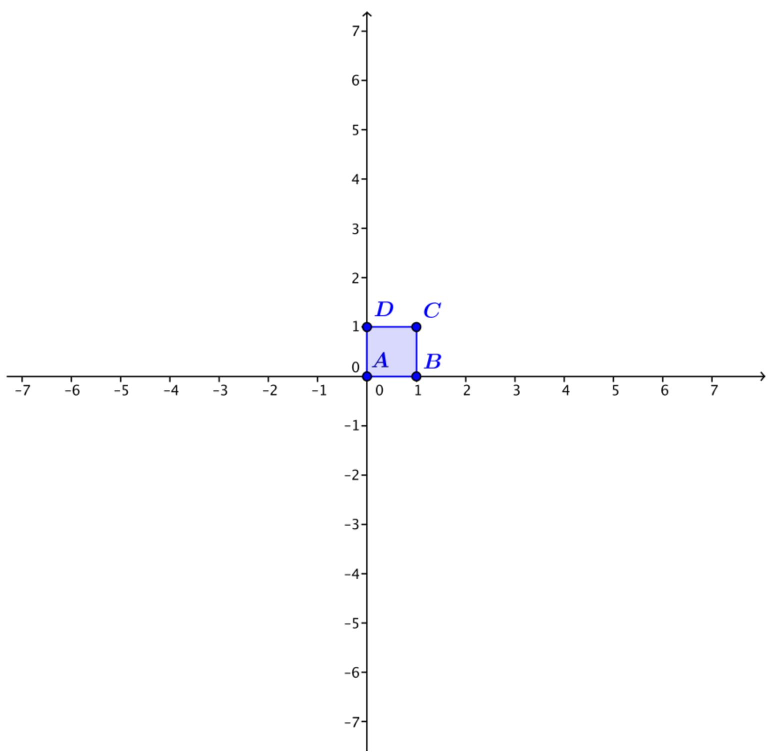
a. Describe the amount the square has been rotated counterclockwise.

b. What is the dilation factor of the square? Explain how you know.

c. What is the geometric effect of your transformation L_1, L_2, L_3, or L_4 on the unit square $ABCD$?

d. Make a conjecture: What do you expect to be the geometric effect of the transformation $L(z) = (2 + i)z$ on the unit square $ABCD$?

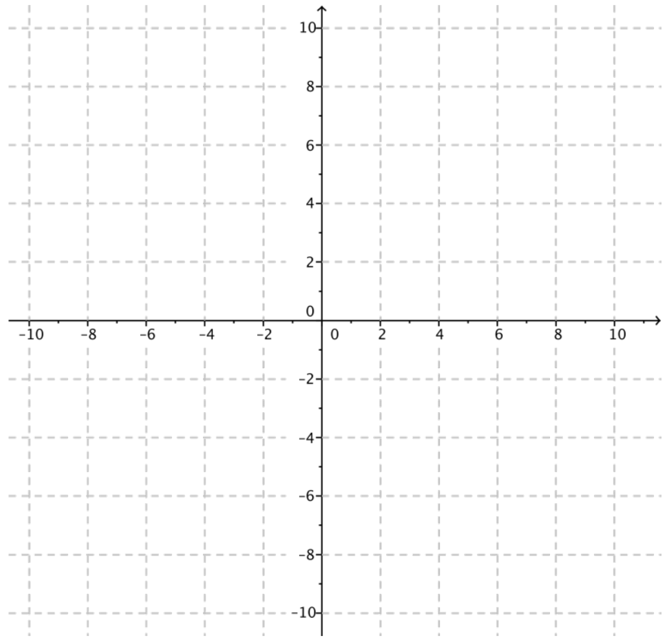
e. Test your conjecture with the unit square on the axes below.



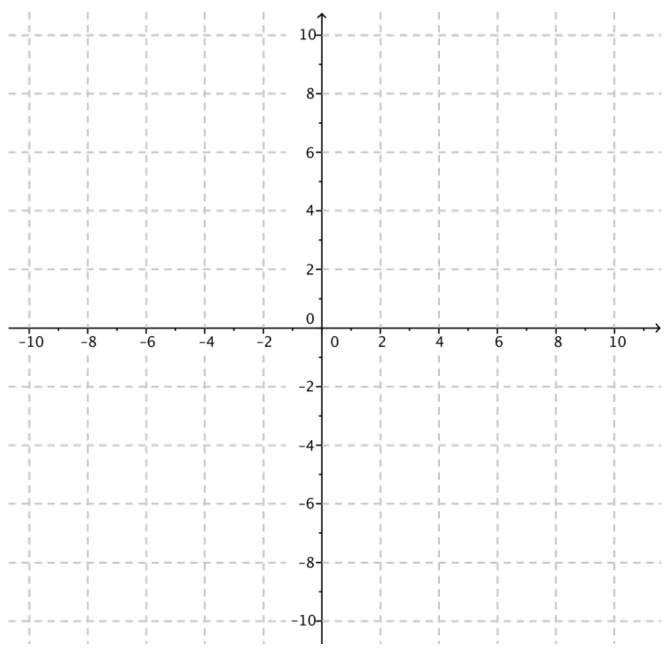
IV. Applications

11. For each exercise below, compute the product wz. Then, plot the complex numbers z, w, and wz on the axes provided.

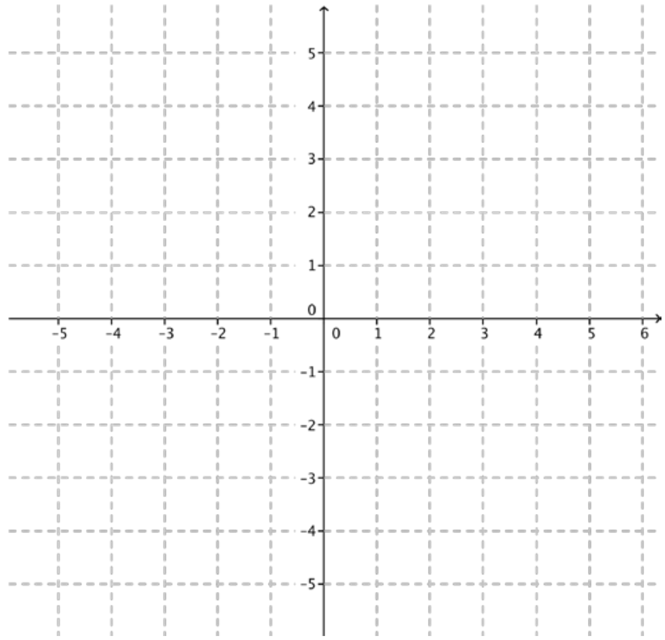
a. $z = 3 + i, w = 1 + 2i$



b. $z = 1 + 2i, w = -1 + 4i$



c. $z = -1 + i, w = -2 - i$



d. For each part (a), (b), and (c), draw line segments connecting each point z, w and wz to the origin. Determine a relationship between the arguments of the complex numbers z, w, and wz.

12. Let $w = a + bi$ and $z = c + di$.

a. Calculate the product wz.

b. Calculate the moduli $|w|$, $|z|$, and $|wz|$.

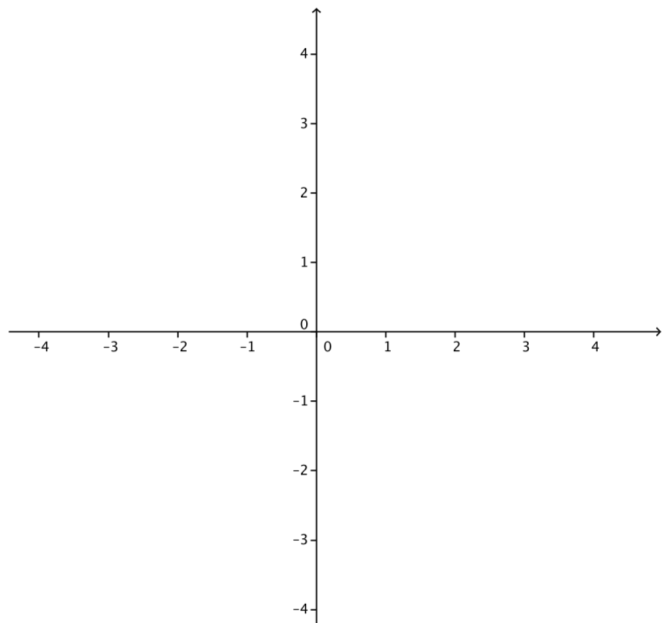
c. What can you conclude about the quantities $|w|$, $|z|$, and $|wz|$?

13. What does the result of Questions 11-12 tell us about the geometric effect of the transformation $L(z) = wz$?

14. If z and w are the complex numbers with the specified arguments and moduli, locate the point that represents the product wz on the provided coordinate axes.

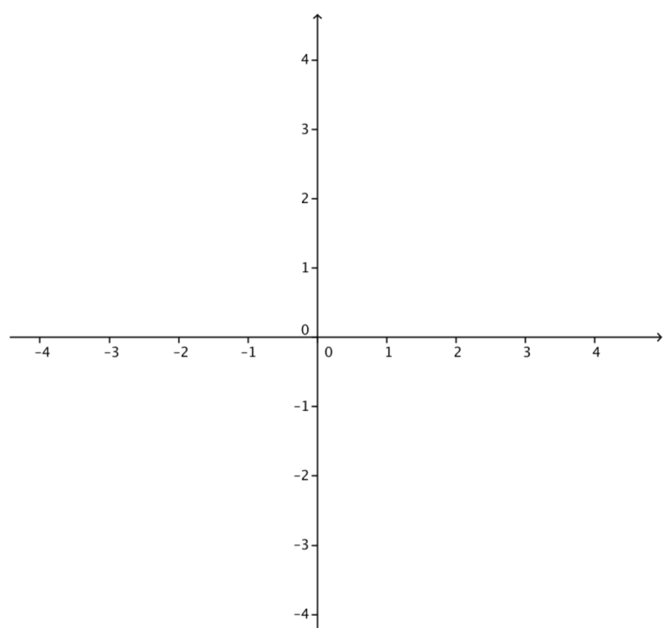
a. $|w| = 3,\ \arg(w) = \frac{\pi}{4}$

$|z| = \frac{2}{3},\ \arg(z) = -\frac{\pi}{2}$



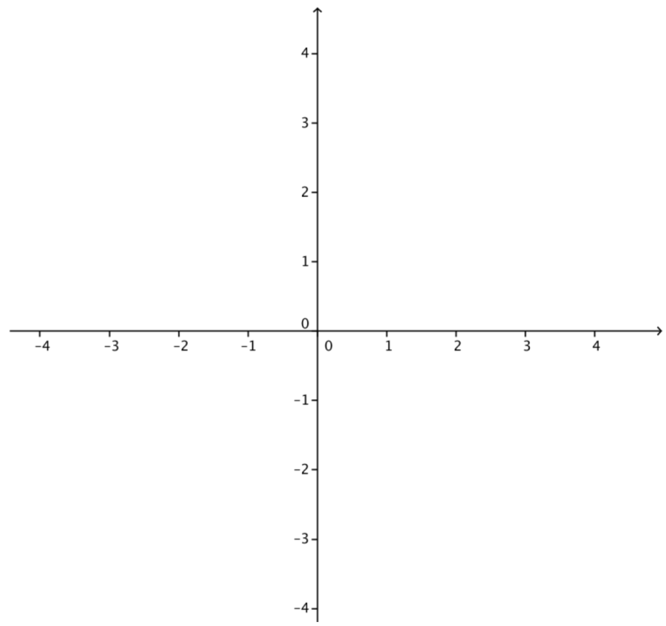
b. $|w| = 2,\ \arg(w) = \pi$

$|z| = 1,\ \arg(z) = \frac{\pi}{4}$



c. $|w| = \frac{1}{2},\ \arg(w) = \frac{4\pi}{3}$

$|z| = 4,\ \arg(z) = -\frac{\pi}{6}$



Lesson Summary

For complex numbers z and w,

- The modulus of the product is the product of the moduli:

$$|wz| = |w| \cdot |z|,$$

- The argument of the product is the sum of the arguments:

$$\arg(wz) = \arg(w) + \arg(z).$$