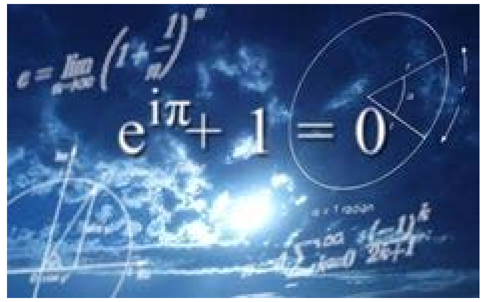
5.5: Polar Form and DeMoivre's Theorem

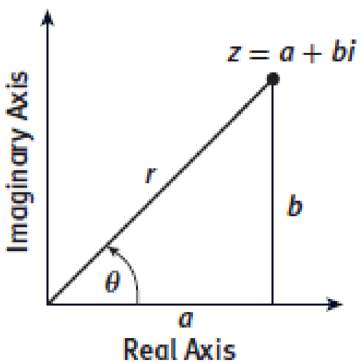
Complex Roots



The **polar form** of a complex number z uses trigonometry to express a complex number in terms of its modulus $|z|$ and argument $\theta = \arg(z)$.

In the diagram shown below, $r = |z|$, and θ is the smallest positive angle formed by the positive real axis and the modulus of z.

1. Use the diagram below to answer the following.



a. Express r in terms of a and b.

b. Use trigonometry to express a, the real part of z, in terms of r and θ.

c. Use trigonometry to express b, the imaginary part of z, in terms of r and θ.

d. Write the polar form of $z = a + bi$ by writing it in terms of r and θ.

2. Complete the table below for each complex number.

z	$r = \|z\|$	θ	Polar Form of z
i	1	90°	$1(\cos(90°) + i\sin(90°))$
$3i$			
$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$			
$-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$			
$-2 + 2i$			

3. In the table below, z is the product of two complex numbers listed in the table for Item 20. Complete the table for each product by determining r and θ and expressing each product in polar form.

z	$r = \|z\|$	θ	Polar Form of z
$i \cdot \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$			
$3i \cdot \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$			
$i \cdot (-2 + 2i)$			
$3i \cdot (-2 + 2i)$			

4. Compare the values in the tables for Items 20 and 21, and describe any patterns that you notice about the angle and the r-value of the products and factors.

5. Multiplying the polar forms of two complex numbers provides insight into the geometric properties of complex multiplication.

a. Multiply the following general forms of two complex numbers using binomial multiplication.

$[r_1(\cos(\theta_1) + i\sin(\theta_1))] \cdot [r_2(\cos(\theta_2) + i\sin(\theta_2))] =$

b. Write the real part of the result in Part a. What is a trigonometric identity that can be used to simplify this real part?

c. Write the imaginary part of the result in Part a. What is a trigonometric identity that can be used to simplify this imaginary part?

d. Explain how the work in Parts b and c confirms the two properties about the absolute value and argument of the product of complex numbers.

II. DeMoivre's Theorem

If a complex number $z = a + bi$ is written in polar form as $z = r(\cos(\theta) + i\sin(\theta))$ then

$z^n = [r(\cos(\theta) + i\sin(\theta))]^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

This formula is known as **De Moivre's Theorem**.

DE MOIVRE'S THEOREM

If $z = r(\cos\theta + i\sin\theta)$, then for any integer n

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

6. Review your work in the Activity and cite specific discoveries that make the above theorem plausible.

7. Use De Moivre's Theorem to find each of the following.

a. $[3(\cos(60°) + i\sin(60°))]^6$

b. $(-2i)^{10}$

c. $(-\sqrt{3} + i)^4$

d. $(2 - 2i)^5$

An ***n*th root** of a complex number z is any complex number w such that w^n=z. De Moivre's Theorem gives us a method for calculating the nth roots of any complex number.

nth ROOTS OF COMPLEX NUMBERS

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then z has the n distinct nth roots

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n-1$.

The following observations help us use the preceding formula.

FINDING THE nth ROOTS OF $z = r(\cos\theta + i\sin\theta)$

1. The modulus of each nth root is $r^{1/n}$.
2. The argument of the first root is θ/n.
3. We repeatedly add $2\pi/n$ to get the argument of each successive root.

8. Find the six sixth roots of z = -64, and graph these roots in the complex plane.