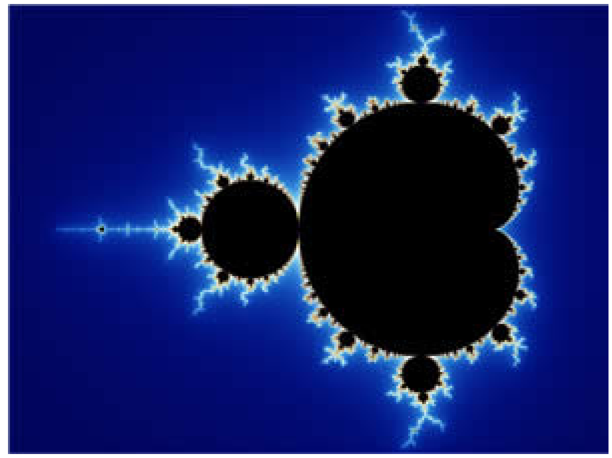
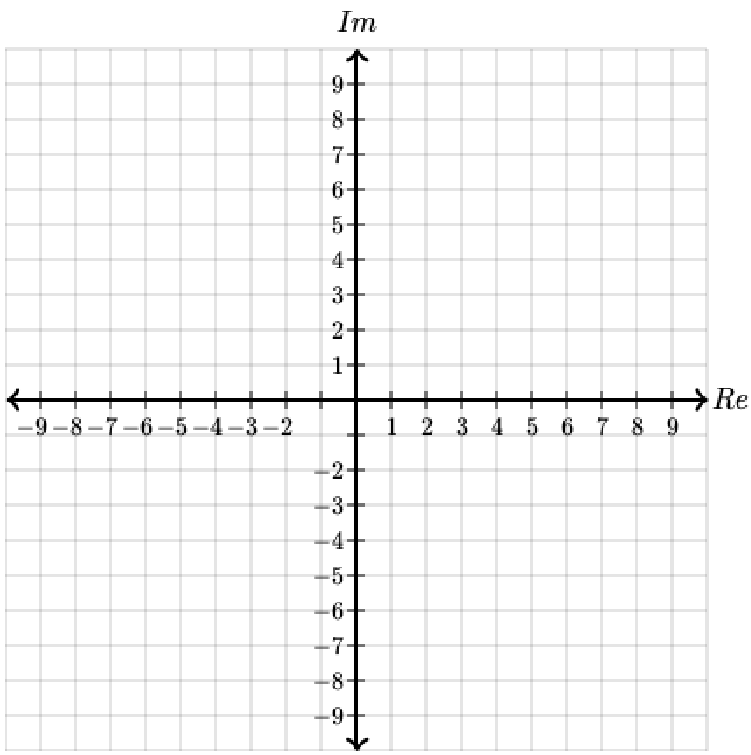
5.3: Complex Multiplication

The Geometric Effect of Complex Multiplication



In Lesson 2.2, you explored the geometric effects of adding and subtracting complex numbers. In this lesson, you will explore the geometric effect on multiplication of complex numbers.



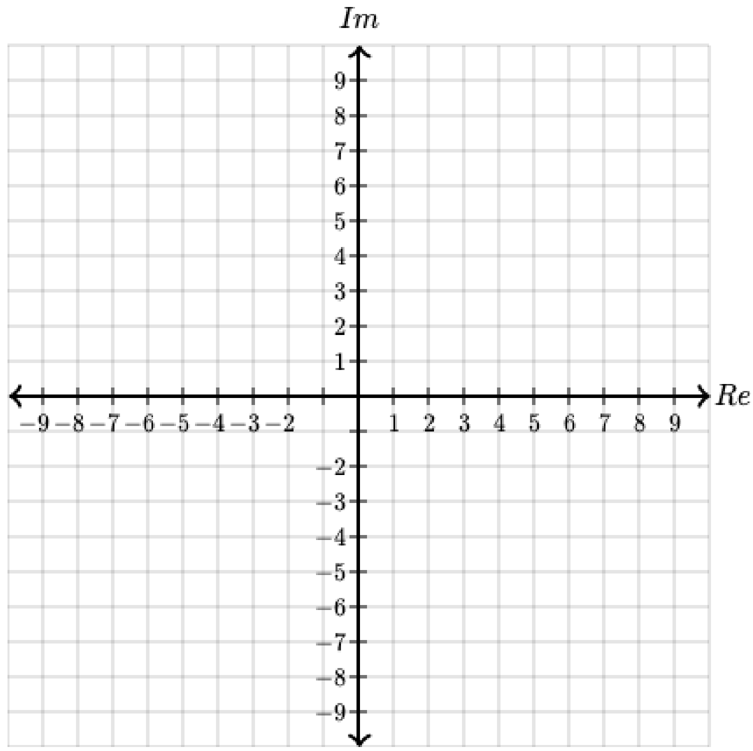
1. Plot the given points, then plot the image $L(z) = 2z$. Label the original points with capital letters (a, etc.)and the transformed image with primes (A', etc.)

   a. $z_1 = 3$
   b. $z_2 = 2i$
   c. $z_3 = 1 + i$
   d. $z_4 = -4 + 3i$
   e. $z_5 = 2 - 5i$

   f. What is the geometric effect on a complex number (the transformation) when you multiply by a Real number?

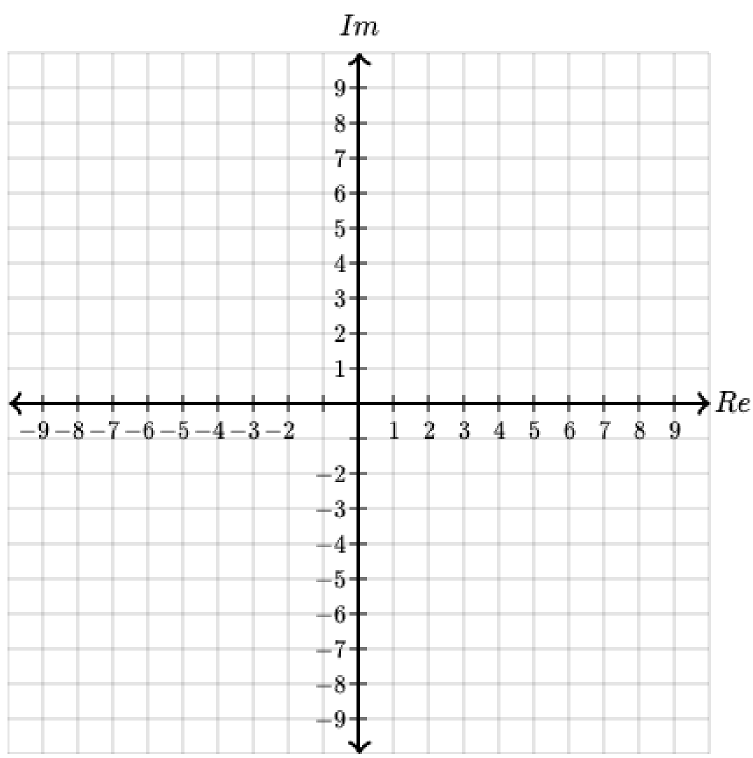
2. Plot the given points, then plot the image $L(z) = iz$.
   a. $z_1 = 3$
   b. $z_2 = 2i$
   c. $z_3 = 1 + i$
   d. $z_4 = -4 + 3i$
   e. $z_5 = 2 - 5i$

   f. What is the geometric effect of the transformation? Confirm your conjecture using the slope of the segment joining the origin to the point and then to its image.



3. Describe the geometric effect of $L(z) = (1 + i)z$ given the following. Plot the images on graph paper, and describe the geometric effect in words.
   a. $z_1 = 1$
   b. $z_2 = i$
   c. $z_3 = 1 + i$
   d. $z_4 = 4 + 6i$

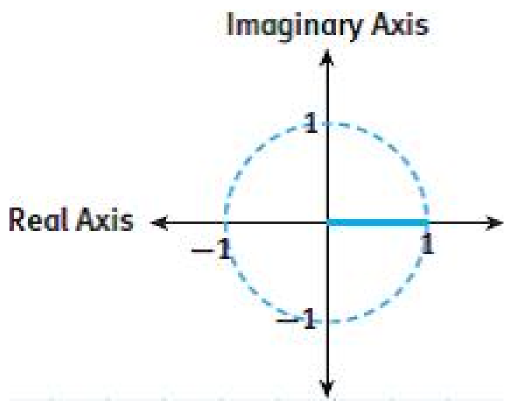
   e. What is the geometric effect of the transformation?



II. Multiplying Complex Numbers and Graphing the Products

Investigate the effects of multiplying complex numbers whose *modulus* is 1.

4. Explore the impact of repeatedly multiplying the complex number 1 by the complex number i. Find the following products and then plot the results on the right. For each product, draw the *modulus*. The number 1 has been plotted.



a. $1 \cdot i =$

b. $1 \cdot i \cdot i =$

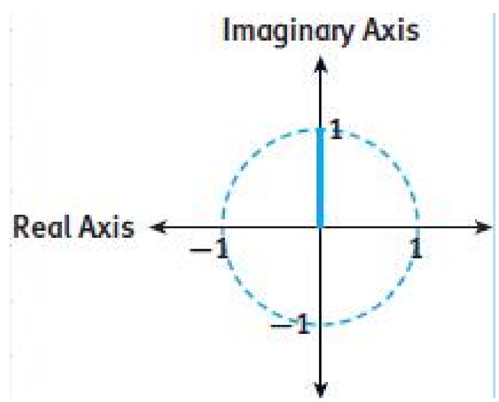
c. $1 \cdot i \cdot i \cdot i =$

d. $1 \cdot i \cdot i \cdot i \cdot i =$

e. $1 \cdot i \cdot i \cdot i \cdot i \cdot i =$

f. Summarize what will happen if you continue multiplying by i.

5. Explore the impact of repeatedly multiplying the complex number i by the complex number i. Find the following products and then plot the results on the right. For each product, draw the modulus . The number i has been plotted.



a. $i \cdot i =$

b. $i \cdot i \cdot i =$

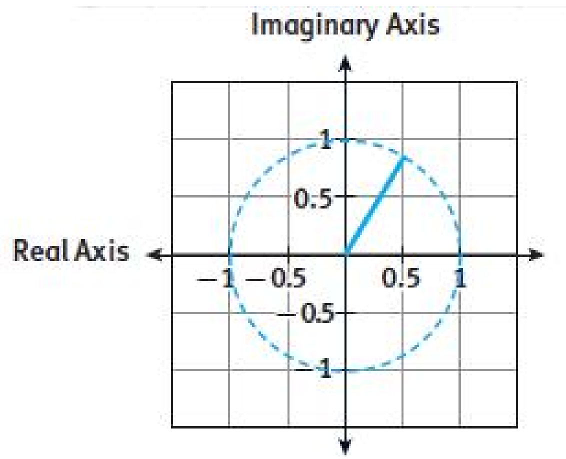
c. $i \cdot i \cdot i \cdot i =$

d. $i \cdot i \cdot i \cdot i \cdot i =$

e. $i \cdot i \cdot i \cdot i \cdot i \cdot i =$

f. Summarize what will happen if you continue multiplying by *i*.

6. Explore the impact of repeatedly multiplying the complex number $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ by the complex number *i*. Find the following products and then plot the results. For each product, draw the modulus. The number $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ has been plotted.



a. $i\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) =$

b. $i \cdot i\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) =$

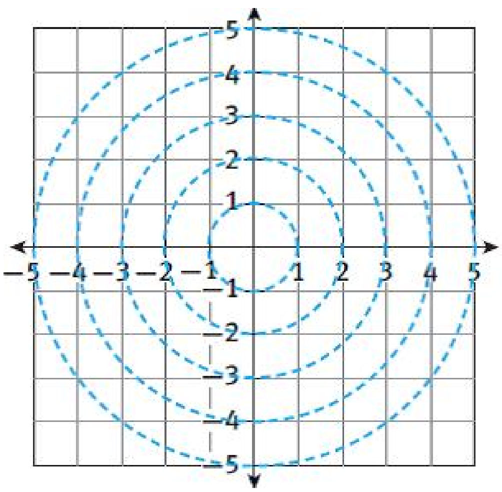
c. $i \cdot i \cdot i\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) =$

d. $i \cdot i \cdot i \cdot i\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) =$

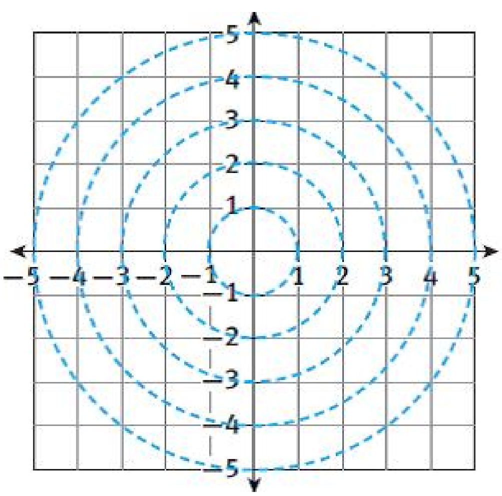
7. Use your results in Items 5–7 to answer the following.

a. Describe the geometric pattern of repeatedly multiplying the numbers 1, *i*, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ by the complex number *i*.

b. State a hypothesis about multiplying a complex number by *i*. Then choose a complex number other than 1, *i*, or $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and test your hypothesis. A complex plane has been provided.

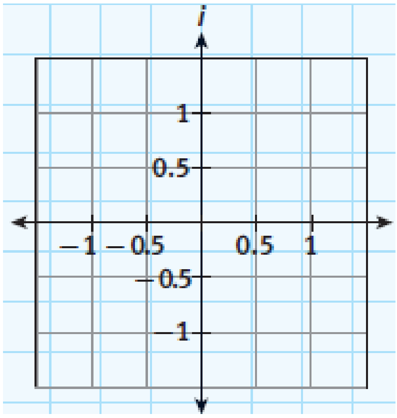


c. Plot the complex number i on the complex plane below. Is there a geometric property about the location of i that indicates why the overall effect of multiplication by i should be as you describe in Part b? Explain.



8. Investigate multiplication by a complex number with an argument other than $\frac{\pi}{2}$ radians (or 90°). Let w denote the complex number $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

a. Find $|w|$ and $\arg(w)$ and plot w on the complex plane below. Draw the modulus.



b. Compute the following products and then plot the results. For each product, draw the segment from zero to the number. Label each plotted point.

$w^2 = w \cdot w =$

$w^3 = w \cdot w \cdot w =$

$w^4 = w \cdot w \cdot w \cdot w =$

c. Describe the effect that repeated multiplication by $w = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$ has on the number w.

d. Use your answer in Part c to predict the solutions to the following.

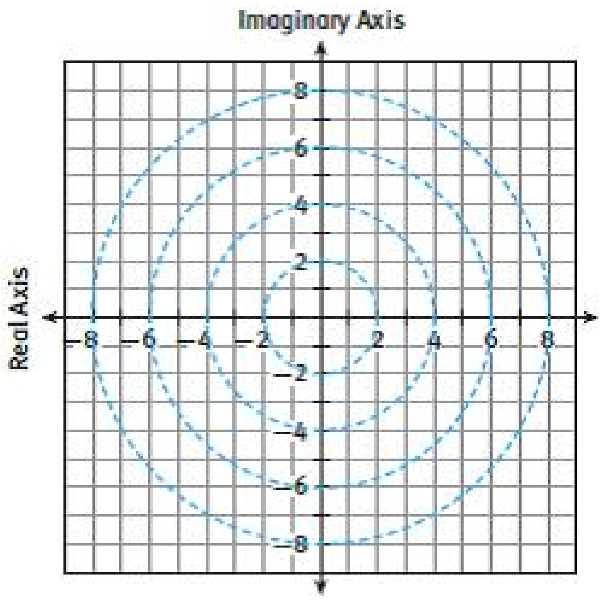
$w^5 =$

$w^6 =$

$w^7 =$

$w^8 =$

9. Now investigate the effects of multiplying complex numbers whose modulus (absolute value) is a number other than 1. Let $z = \sqrt{2} + \sqrt{2}i$



a. Find $|z|$ and $\arg(z)$. Then plot the complex number z.

b. Calculate z^2 and z^3 .

c. Determine the absolute value and argument of z^2 and z^3. Then plot these complex numbers and draw the modulus.

10. Consider the two complex numbers $w = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and $z = \sqrt{2} + \sqrt{2}i$

a. Compare and contrast the two complex numbers.

b. Compare and contrast the products z^2 and z^3 to the products w^2 and w^3 .

11. Find the following arguments, where $w = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

a. $\arg(w) =$

b. $\arg(\ w^2\ ) =$

c. $\arg(\ w^3\ ) =$

d. $\arg(\ w^4\ ) =$

12. Find the following arguments, where $z = \sqrt{2} + \sqrt{2}i$

a. $\arg(z) =$

b. $\arg(z^2) =$

c. $\arg(z^3) =$

d. $\arg(z^4) =$

13. Based on your responses in Items 12 and 13, how is $\arg(z)$ related to $\arg(z^n)$? Explain.

14. Let $z = 2\sqrt{3} + 2i$ and $w = 1 + \sqrt{3}i$

a. Find the moduli (plural of *modulus*) of <u>both</u> z and w.

b. Find the product of $z \cdot w$.

c. Find the modulus of $|z \cdot w|$.

d. What is the relationship between the modulus of the product of $z \cdot w$ and the individual moduli of z and w?

15. Find the following arguments.

a. $\arg(2\sqrt{3} + 2i)$

b. $\arg(1 + \sqrt{3}i)$

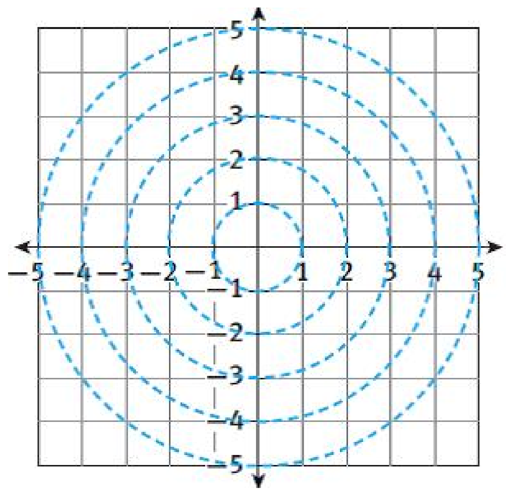
c. $\arg[(2\sqrt{3} + 2i) \cdot (1 + \sqrt{3}i)]$

16. Based on your response to Item 15, how are the arguments of two complex numbers, w and z, related to the argument of their product, $\arg(w \cdot z)$?

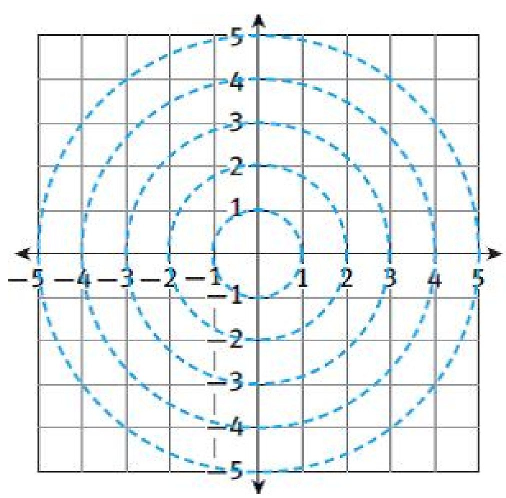
17. For each product below, plot each factor on its own complex plane. Next, predict the

location of the corresponding product of these factors on the complex plane using your conclusions from Items 10–17. Finally, test your prediction by finding the product algebraically, and determining its absolute value and argument.

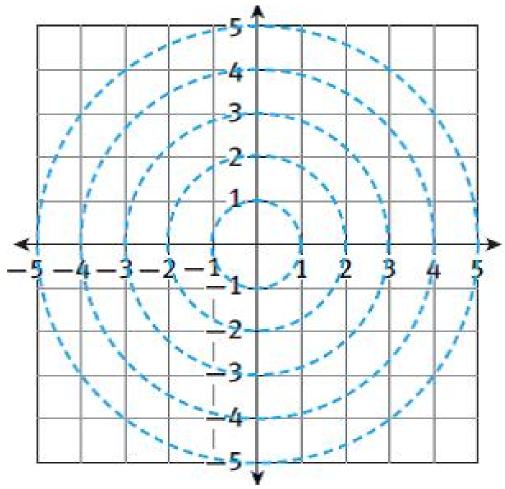
a. $(i)(2 + 2i)$



b. $(-i)(2 + 2i)$



c. $(1 + i)(2 + 2i)$



d. $(i)(2\sqrt{3} + i)$