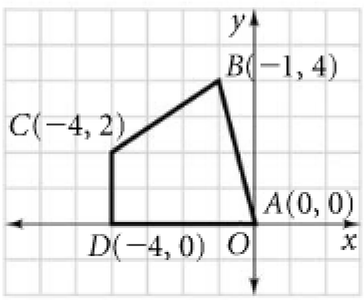
4.9: Matrices as Transformations of the Coordinate Plane

Geometric Transformations

Graphic design and 3-D Modeling software programs use matrix mathematics to process transformations and render images. A square matrix, one with exactly as many rows as columns, can represent a linear transformation of a geometric object. For example, in the Cartesian X-Y plane, the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ reflects an object over the vertical Y axis. In a video game, this would render the upside-down mirror image of a castle reflected in a lake.

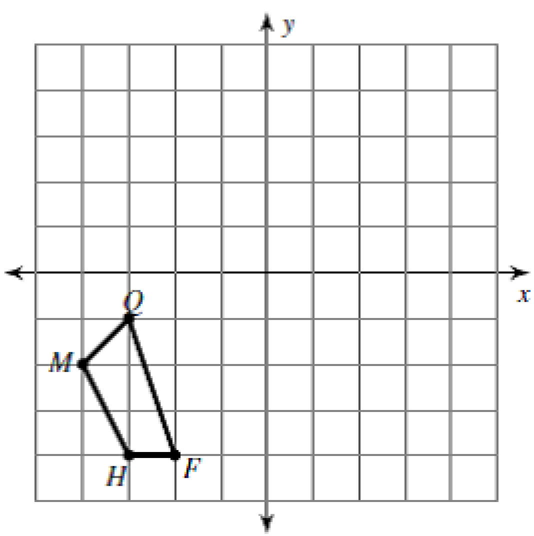
We can write the vertices of a figure as a matrix. For example the matrix



$$\begin{matrix} & A & B & C & D \\ x\text{-coordinate} & \left[\begin{matrix} 0 \\ 0 \end{matrix}\right. & \begin{matrix} -1 \\ 4 \end{matrix} & \begin{matrix} -4 \\ 2 \end{matrix} & \left.\begin{matrix} -4 \\ 0 \end{matrix}\right] \end{matrix}$$

	A	B	C	D
x-coordinate	0	−1	−4	−4
y-coordinate	0	4	2	0

1. Write a coordinate matrix for the vertices of polygon *FHMQ*.



II. Translating a 2-Dimensional Figure

Now that we can represent a two-dimensional object in a matrix, we can transform them in the coordinate plane. First we will translate these matrices. A **translation** is a vertical or horizontal shift of the vector within the plane. We can accomplish this through matrix addition/subtraction.

To translate a matrix, we will create a **translation matrix** and add this to the coordinate matrix to create a new matrix that contains the translated coordinates.

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} + \begin{bmatrix} \text{Horiz. Trans.} & \text{Horiz. Trans} \\ \text{Vert. Trans.} & \text{Vert. Trans.} \end{bmatrix} = \begin{bmatrix} x_1' & x_2' \\ y_1' & y_2' \end{bmatrix} = (x_1', y_1'), (x_2', y_2')$$

EXAMPLE 1 – Translating a 2-D Figure

Quadrilateral ABCD has vertices A(0,0), B(-1,4), C(-4,2) and D(-4,0). Use a matrix to find the coordinates that are translated 6 units right and 2 units down. Graph $ABCD$ and its image $A'B'C'D'$.

SOLUTION:

Vertices of the Quadrilateral | **Translation Matrix** | **Vertices of the Image**

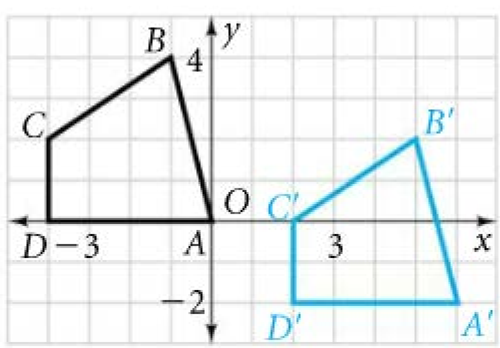
Add 6 to each *x*-coordinate.

$$\begin{array}{cccc} A & B & C & D \end{array} \qquad \qquad \qquad \qquad \begin{array}{cccc} A' & B' & C' & D' \end{array}$$

$$\begin{bmatrix} 0 & -1 & -4 & -4 \\ 0 & 4 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 6 & 6 & 6 \\ -2 & -2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 5 & 2 & 2 \\ -2 & 2 & 0 & -2 \end{bmatrix}$$

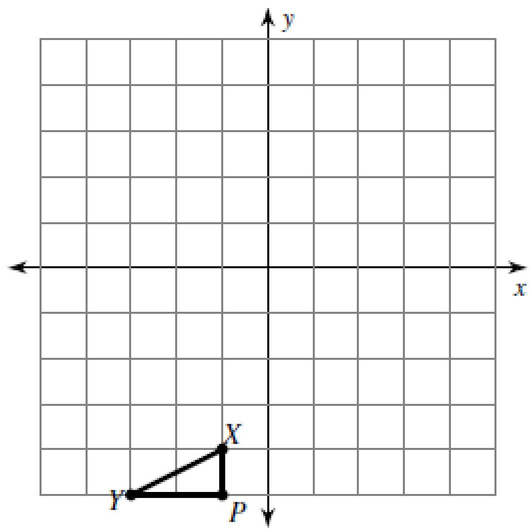
Subtract 2 from each *y*-coordinate.

So, the vertices are A’(6,-2), B’(5,2), C’(2,0) and D’(2.-2), and the graphs are below.



Your Turn:

2. Use a matrix to find the coordinates of ΔPXY that are translated 3 units right and 5 units up. Graph the translated image.



3. Write a translation matrix to translate the vertices of a pentagon 3 units left and 2 units up?

4. Use your answer to question 3 to translate a pentagon with vertices A(0,-5), B(-1,-1), C(5,0), D(1,3) and E(4,0). Graph the preimage and image.

III. Reflecting a 2-D Figure

The next transformations of vectors in the coordinate plane we will learn are reflections. A **reflection** is a transformation that maps each point in an object to its mirror image, using a specific line of reflection.

To reflect using a matrix, we will take a **reflection matrix** and multiply it by the coordinate matrix to create a new matrix containing the coordinates of the reflected image. Please note that since matrix multiplication is not commutative, the order matters and the reflection matrix must come first.

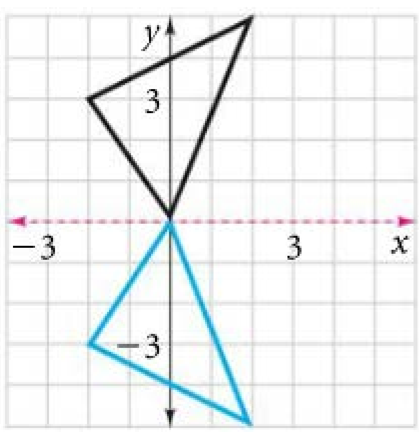
$$[\text{Reflection Matrix}]\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} x_1' & x_2' \\ y_1' & y_2' \end{bmatrix} = (x_1', y_1'), (x_2', y_2')$$

The four most common reflection matrices are:

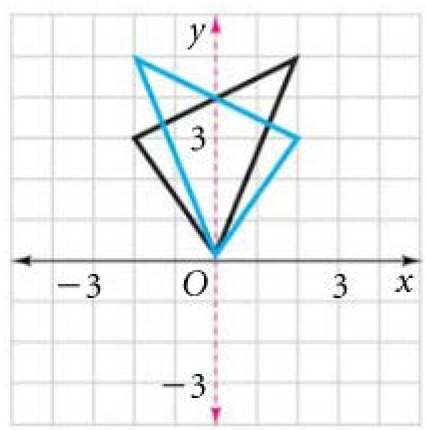
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

5. Identify which of the four reflection matrices listed above create the following transformations of ΔABC. Show your matrix multiplication.

   a. across the x-axis

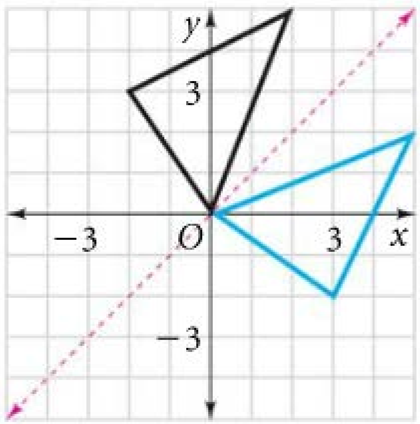
   

   b. across the y-axis

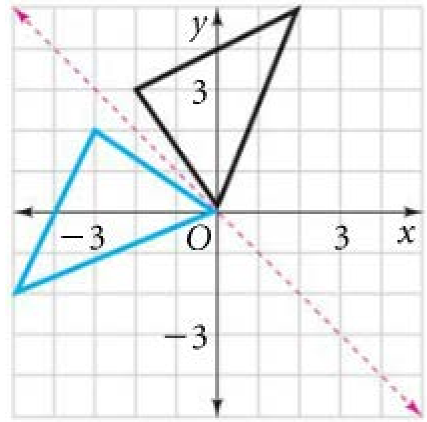
   

6. Identify which of the four reflection matrices listed above create the following transformations of ΔABC. Show your matrix multiplication.

a. across the line y = x



b. across the line y = -x



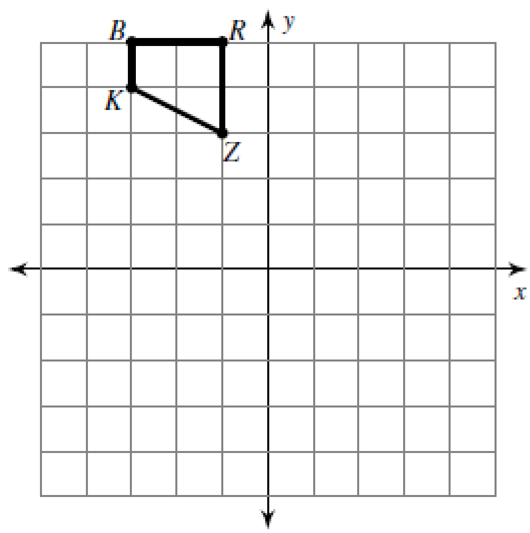
7. Use a matrix to find the coordinates of quadrilateral *BRKZ* that are reflected over the x-axis, the y-axis, and the lines $y = \pm x$. Graph the three translated images.

a. across the x-axis :

b. across the y-axis:

c. across the line y = x:

d. across the line y = -x:

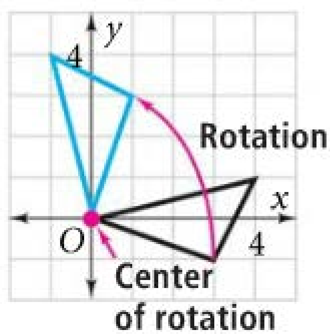


IV. Rotating a 2-D Figure

A **rotation** is a transformation that turns a figure around a fixed point called the center of rotation. To rotate using a matrix, we will take a rotation matrix and multiply it by our coordibnate matrix to create a new matrix that contains the translated coordinates.

$$[\text{Rotation Matrix}]\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} x_1' & x_2' \\ y_1' & y_2' \end{bmatrix} = (x_1', y_1'), (x_2', y_2')$$

As you recall from trigonometry, angles of rotation are measured counterclockwise about the origin.



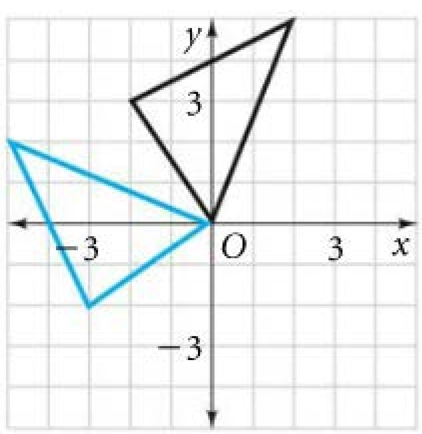
The three most common rotation matrices are:

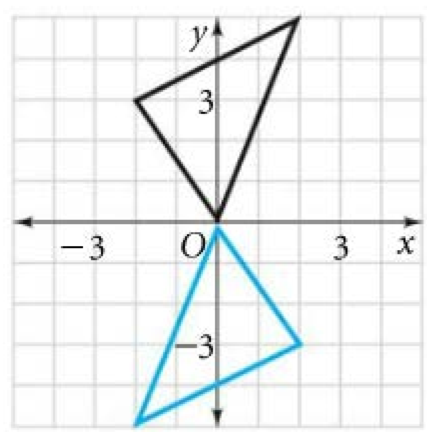
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

8. Identify which of the four reflection matrices listed above create the following transformations of ΔABC. Show your matrix multiplication.

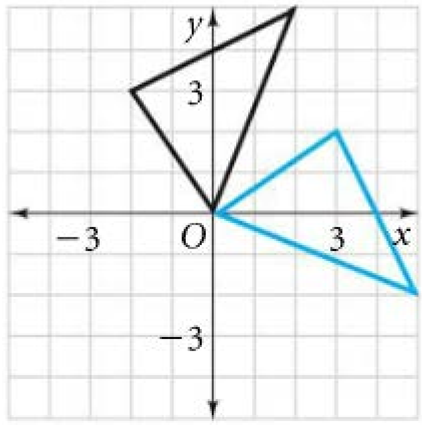
a. 90º

b. 180º





c. 270°

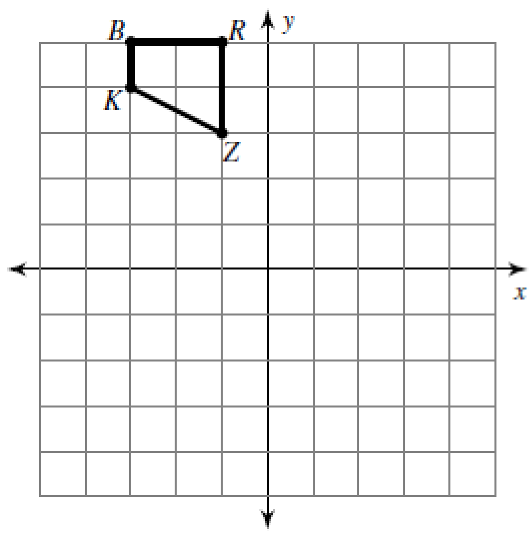


9. Use a matrix to find the coordinates of quadrilateral *BRKZ* after rotations of 90°, 180°, and 270°. Graph the three translated images.

   a. 90° :

   b. 180° :

   c. 270° :



10. Write the rotation matrix for a 360° rotation.

V. Dilating a 2-D Figure

The final transformations in the coordinate plane are dilations. A **dilation** is a transformation where the shape of an object remains the same, but its size is changed, either made larger or smaller. To dilate using a matrix, you will perform scalar multiplication.

$$[\text{Scale Factor}]\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} x_1' & x_2' \\ y_1' & y_2' \end{bmatrix} = (x_1', y_1'), (x_2', y_2')$$

11. Use a dilation matrix to find the coordinates of ΔFJT after it is dilated by the following scale factors. Then graph the three dilated images.

    a. 3

    b. $\frac{1}{2}$