# 4.5: Nutritional Analysis

*Linear Systems and Matrices*

A nutritionist is performing an experiment on the effects of various combinations of vitamins. She wishes to feed each of her laboratory rabbits a diet that contains exactly 5 mg of niacin and 18 mg of riboflavin. She has available two different types of commercial rabbit pellets; their vitamin content (per ounce) is given in the table. How many ounces of each type of food should each rabbit be given daily to satisfy the experiment requirements?

| Vitamin Content | Boost-X | Y-Not |
|---|---|---|
| Niacin (mg) | 1 | 2 |
| Riboflavin (mg) | 5 | 3 |

1. Write two linear equations that represent the unknown Boost-X and Y-not quantities combining to equal the total desired vitamins.

2. Graph the two equations on the same coordinate system and interpret the solution.

3. Solve the system of equations using the Substitution Method

## II. Linear Systems in Three Variables

The following are two examples of systems of linear equations in three variables. The second system is in **triangular form** (aka. **row-echelon** form); that is, the variable *x* doesn't appear in the second equation, and the variables *x* and *y* do not appear in the third equation.

**A system of linear equations**

$$\begin{cases} x - 2y - z = 1 \\ -x + 3y + 3z = 4 \\ 2x - 3y + z = 10 \end{cases}$$

**A system in triangular form**

$$\begin{cases} x - 2y - z = 1 \\ y + 2z = 5 \\ z = 3 \end{cases}$$

It's easy to solve a system that is in triangular form by using back-substitution.

4. $$\begin{cases} x + y - 3z = 8 \\ y - 3z = 5 \\ z = -1 \end{cases}$$

   a. From the last equation, we know z = -1. Back-substitute this value in Equation 2 and solve for *y*.

   b. Back-substitute the values for *y* and *z* into the first equation and solve for *x*.

<u>Your turn</u>: Use back-substitution to solve the following triangular systems:

c. $$\begin{cases} x - 2y + 3z = 10 \\ 2y - z = 2 \\ 3z = 12 \end{cases}$$

d. $$\begin{cases} 4x + 3z = 10 \\ 2y - z = -6 \\ \frac{1}{2}z = 4 \end{cases}$$

## III. Gaussian Elimination

To change a system of linear equations to an **equivalent system** (that is, a system with the same solutions as the original system), we use the elimination method. This means that we can use the following operations.

> **OPERATIONS THAT YIELD AN EQUIVALENT SYSTEM**
>
> 1. Interchange the positions of two equations.
> 2. Add a nonzero multiple of one equation to another.
> 3. Multiply an equation by a nonzero constant.

To solve a linear system, we use these operations to change the system to an equivalent triangular system. Then we use back-substitution as in Example 1. This process is called **Gaussian elimination.**

EXAMPLE 1 - Solve the system of Linear Equations:

$$\begin{cases} x - 2y + 3z = 9 & \text{Equation 1} \\ -x + 3y \qquad = -4 & \text{Equation 2} \\ 2x - 5y + 5z = 17 & \text{Equation 3} \end{cases}$$

Because the leading coefficient of the first equation is 1, we can leave Equation 1 as it is.

$$\begin{array}{rll} x - 2y + 3z = & 9 & \text{Write Equation 1.} \\ -x + 3y \qquad = & -4 & \text{Write Equation 2.} \\ \hline y + 3z = & 5 & \text{Add Equation 1 to Equation 2.} \end{array}$$

$$\begin{cases} x - 2y + 3z = 9 \\ \qquad y + 3z = 5 \\ 2x - 5y + 5z = 17 \end{cases}$$

Adding the first equation to the second equation produces a new second equation.

$$\begin{array}{rll} -2x + 4y - 6z = & -18 & \text{Multiply Equation 1 by } -2. \\ 2x - 5y + 5z = & 17 & \text{Write Equation 3.} \\ \hline -y - z = & -1 & \text{Add revised Equation 1 to Equation 3.} \end{array}$$

$$\begin{cases} x - 2y + 3z = 9 \\ \qquad y + 3z = 5 \\ \quad -y - z = -1 \end{cases}$$

Adding $-2$ times the first equation to the third equation produces a new third equation.

Now that all but the first have been eliminated from the first column, we will work on the second column. (We need to eliminate $y$ from the third equation.)

$$\begin{cases} x - 2y + 3z = 9 \\ \quad\;\; y + 3z = 5 \\ \qquad\quad 2z = 4 \end{cases}$$

Adding the second equation to the third equation produces a new third equation.

Finally, we need a coefficient of 1 for $z$ in the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ \quad\;\; y + 3z = 5 \\ \qquad\quad\; z = 2 \end{cases}$$

Multiplying the third equation by $\frac{1}{2}$ produces a new third equation.

5. Use back-substitution to find the values of x, y and z.

6. Perform row operations on the following system.

$$\begin{cases} x - 2y + 3z = 5 & \text{Equation 1} \\ -x + 3y - 5z = 4 & \text{Equation 2} \\ 2x \qquad\; - 3z = 0 & \text{Equation 3} \end{cases}$$

   a. Add Equation 1 to Equation 2.

      i. What did this operation accomplish?

      ii. Write the equivalent system

   b. Add -2 times Equation 1 to Equation 3.

      i. What did this operation accomplish?

      ii. Write the equivalent system.

7. Solve the system of linear equations using Gaussian Elimination.

a. $$\begin{cases} x + y + z = 7 \\ 2x - y + z = 9 \\ 3x \quad\; - z = 10 \end{cases}$$

## IV. Solving Systems of Equations with Matrices

You can write a system of linear equations as a matrix, called the **augmented matrix** of the system, by writing only the coefficients and constants that appear in the equations. Here is an example:

| Linear system | Augmented matrix |
|---|---|
| $\begin{cases} 3x - 2y + z = 5 \\ x + 3y - z = 0 \\ -x + \quad 4z = 11 \end{cases}$ | $\begin{bmatrix} 3 & -2 & 1 & 5 \\ 1 & 3 & -1 & 0 \\ -1 & 0 & 4 & 11 \end{bmatrix}$ |

Note the use of 0 for the missing coefficient of the $y$-variable in the third equation.

8. Write the augmented matrix of the system of equations

a. $$\begin{cases} x + y + z = 4 \\ -x + 2y + 3z = 17 \\ 2x - y \quad = -7 \end{cases}$$

b. $$\begin{cases} 6x - 2y - z = 4 \\ x + 3z = 1 \\ 7y + z = 5 \end{cases}$$

To solve a system of linear equations using its augmented matrix, we use elementary row operations to arrive at a matrix in a row-echelon (triangular) form which is even easier to see in matrices.

**Not in row-echelon form**

$$\begin{bmatrix} 0 & 1 & -\frac{1}{2} & 0 & 6 \\ 1 & 0 & 3 & 4 & -5 \\ 0 & 0 & 0 & 1 & 0.4 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Leading 1's do *not* shift to the right in successive rows

**Row-echelon form**

$$\begin{bmatrix} 1 & 3 & -6 & 10 & 0 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's shift to the right in successive rows

Here is a systematic way to put a matrix in row-echelon form using elementary row operations:

- Start by obtaining 1 in the top left corner. Then obtain zeros below that 1 by adding appropriate multiples of the first row to the rows below it.
- Next, obtain a leading 1 in the next row, and then obtain zeros below that 1.
- At each stage make sure that every leading entry is to the right of the leading entry in the row above it—rearrange the rows if necessary.
- Continue this process until you arrive at a matrix in row-echelon form.

This is how the process might work for a 3 x 4 matrix:

$$\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \end{bmatrix} \quad \begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare & \blacksquare \end{bmatrix} \quad \begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix}$$

Once an augmented matrix is in row-echelon form, we can solve the corresponding linear system using back-substitution.

9. In your own words, describe what row-echelon form looks like in a matrix.

EXAMPLE 2 – Solving a System Using Row-Echelon Form

Solve the system of linear equations using Gaussian elimination:

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

First, we write the augmented matrix of the system, and then we use elementary row operations to put it in row-echelon form.

Need a 1 here

Augmented matrix:

$$\left[\begin{array}{rrr|r} 4 & 8 & -4 & 4 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{array}\right]$$

Need 0's here

$$\xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{array}\right]$$

Need a 1 here

$$\xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - 3R_1 \to R_2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 2 & 8 & -14 \\ 0 & 5 & 10 & -15 \end{array}\right]$$

Need a 0 here

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 5 & 10 & -15 \end{array}\right]$$

Need a 1 here

$$\xrightarrow{R_3 - 5R_2 \to R_3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & -10 & 20 \end{array}\right]$$

Row-echelon form:

$$\xrightarrow{-\frac{1}{10}R_3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

10. You now have an equivalent matrix in row-echelon form.
    a. Write the corresponding system of equations.

b. Use back-substitution to solve the system.

11. Find the solution to the system of equations using Gaussian Elimination:

a. $$\begin{cases} x + y + z = 4 \\ -x + 2y + 3z = 17 \\ 2x - y = -7 \end{cases}$$

b. $$\begin{cases} 2y + z = 4 \\ x + y = 4 \\ 3x + 3y - z = 10 \end{cases}$$

Graphing calculators have a “row-echelon form” command that puts a matrix in row-echelon form. On the TI-nSpire, Select MENU > 7: Matrix & Vector > 4: Row-Echelon Form. (On the TI-84 this command is ref.)

c. Checking your row-echelon forms for part (a) and (b) using a calculator.

If you put the augmented matrix of a linear system in *reduced* row-echelon form, then you don’t need to back-substitute to solve the system!

**Reduced row-echelon form**

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1’s have 0’s above and below them

To put a matrix in reduced row-echelon form, we use the following steps.

- Use the elementary row operations to put the matrix in row-echelon form.

- Obtain zeros above each leading entry by adding multiples of the row containing that entry to the rows above it. Begin with the last leading entry and work up.

Here is how the process works for a 3 x 4 matrix:

$$\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix} \to \begin{bmatrix} 1 & \blacksquare & 0 & \blacksquare \\ 0 & 1 & 0 & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \blacksquare \\ 0 & 1 & 0 & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix}$$

Using the reduced row-echelon form to solve a system is called **Gauss-Jordan elimination**. The process is illustrated in the next example.

EXAMPLE 3 – Solving a System Using Reduced Row-Echelon Form

Solve the system of linear equations, using Gauss-Jordan elimination.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

In Example 2 we used Gaussian elimination on the augmented matrix of this system to arrive at an equivalent matrix in row-echelon form. We continue using elementary row operations on the last matrix in Example 3 to arrive at an equivalent matrix in reduced row-echelon form.

Need 0's here

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Need a 0 here

$$\xrightarrow[R_1 + R_3 \to R_1]{R_2 - 4R_3 \to R_2} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_2 \to R_1} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

12. You now have an equivalent matrix in row-echelon form.
    a. Write the corresponding system of equations.

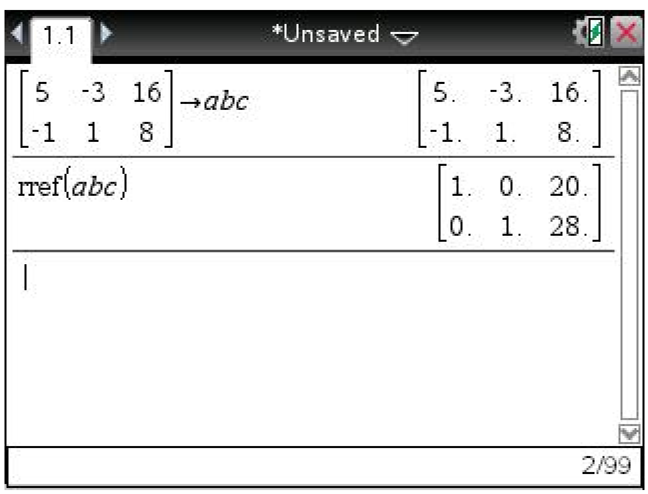
b. What are the values of x, y and z?

13. Find the solution to the system of equations using Gauss-Jordan Elimination:

a. $$\begin{cases} 2x - 3y - z = 13 \\ -x + 2y - 5z = 6 \\ 5x - y - z = 49 \end{cases}$$

b. $$\begin{cases} 10x + 10y - 20z = 60 \\ 15x + 20y + 30z = -25 \\ -5x + 30y - 10z = 45 \end{cases}$$

Graphing calculators have a "reduced row-echelon form". On the TI-nSpire, Select MENU > 7: Matrix & Vector > 5: Reduced Row-Echelon Form. (On the TI-84 this command is rref.)




c. Use a calculator to check your reduced row-echelon form matrices for part (a) and (b).

## V. Inconsistent and Dependent Systems

The systems of linear equations that we considered in Examples 1–3 had exactly one solution. But remember that a linear system may have one solution, no solution, or infinitely many solutions. Fortunately, the row-echelon form of a system allows us to determine which of these cases applies, as illustrated below.

First we need some terminology. A **leading variable** in a linear system is one that corresponds to a leading entry in the row-echelon form of the augmented matrix of the system.

**No solution**

$$\begin{bmatrix} 1 & 2 & 5 & 7 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Last equation says 0 = 1

**One solution**

$$\begin{bmatrix} 1 & 6 & -1 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

Each variable is a leading variable

**Infinitely many solutions**

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$z$ is not a leading variable

Only one of the following applies:

- No solution. If the row-echelon form contains a row that represents the equation $0 = c$, where $c$ is not zero, then the system has no solution. A system with no solution is called **inconsistent**.
- One solution. If each variable in the row-echelon form is a leading variable, then the system has exactly one solution, which we find using back-substitution or Gauss-Jordan elimination.
- Infinitely many solutions. If the variables in the row-echelon form are not all leading variables and if the system is not inconsistent, then it has infinitely many solutions. In this case the system is called **dependent**.

14. Describe the graphs of the following:
    a. a linear system with one solution

    b. an inconsistent linear system

    c. a dependent linear system

If there is a single solution state, you state the values of the variables. If the system is inconsistent, state “no solution. If, however, the linear system is dependent, then we solve the system by putting the matrix in reduced row-echelon form and then expressing the leading variables in terms of the nonleading variables. The nonleading variables may take on any real numbers as their values.

EXAMPLE 4 – A System with Infinitely Many Solutions

Find the complete solution of the system.

$$\begin{cases} x + 2y - 3z - 4w = 10 \\ x + 3y - 3z - 4w = 15 \\ 2x + 2y - 6z - 8w = 10 \end{cases}$$

First, we transform the system into reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & -3 & -4 & 10 \\ 1 & 3 & -3 & -4 & 15 \\ 2 & 2 & -6 & -8 & 10 \end{bmatrix} \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & 2 & -3 & -4 & 10 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & -2 & 0 & 0 & -10 \end{bmatrix}$$

$$\xrightarrow{R_3 + 2R_2 \to R_3} \begin{bmatrix} 1 & 2 & -3 & -4 & 10 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2 \to R_1} \begin{bmatrix} 1 & 0 & -3 & -4 & 0 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in now reduced row-echelon form. Since the last row represents the equation $0 = 0$, we may discard it. So the last matrix corresponds to the system:

$$\begin{cases} x \qquad - 3z - 4w = 0 \\ \qquad y \qquad\qquad\quad = 5 \end{cases}$$

Leading variables

To obtain the complete solution, we solve for the leading variables $x$ and $y$ in terms of the nonleading variables z and w and we let z and w be any real numbers. Thus the complete solution is

$$x = 3s + 4t$$
$$y = 5$$
$$z = s$$
$$w = t$$

where $s$ and $t$ are any Real numbers.

15. Find the complete solution of the following dependent system:

$$\begin{cases} x - y \quad + w = 0 \\ 3x \quad - z + 2w = 0 \\ x - 4y + z + 2w = 0 \end{cases}$$

## VI. Application

16. A nutritionist is performing an experiment on student volunteers. He wishes to feed one of his subjects a daily diet that consists of a combination of three commercial diet foods: MiniCal, LiquiFast, and SlimQuick. For the experiment it is important that the subject consume exactly 500 mg of potassium, 75 g of protein, and 1150 units of vitamin D every day. The amounts of these nutrients in one ounce of each food are given in the table. How many ounces of each food should the subject eat every day to satisfy the nutrient requirements exactly?

| | MiniCal | LiquiFast | SlimQuick |
|---|---|---|---|
| Potassium (mg) | 50 | 75 | 10 |
| Protein (g) | 5 | 10 | 3 |
| Vitamin D (units) | 90 | 100 | 50 |

### ROW-ECHELON FORM AND REDUCED ROW-ECHELON FORM OF A MATRIX

A matrix is in **row-echelon form** if it satisfies the following conditions.

1. The first nonzero number in each row (reading from left to right) is 1. This is called the **leading entry**.
2. The leading entry in each row is to the right of the leading entry in the row immediately above it.
3. All rows consisting entirely of zeros are at the bottom of the matrix.

A matrix is in **reduced row-echelon** form if it is in row-echelon form and also satisfies the following condition.

4. Every number above and below each leading entry is a 0.