Proofs Without Words

A elegantly executed proof is a poem in all but the form in which it is written. 

Morris Kline
(American mathematician and educator; 1908 - 1992)

In mathematics, a proof without words is a proof of an identity or mathematical statement which can be demonstrated as self-evident by a diagram without any accompanying explanatory text. Such proofs can be considered more elegant than more formal and mathematically rigorous proofs due to their self-evident nature. When the diagram demonstrates a particular case of a general statement, to be a proof, it must be generalisable.

The statement that the sum of all positive odd numbers up to $2n - 1$ is a perfect square - more specifically, the perfect square $n^2$—can be demonstrated by a proof without words, as shown on the right. The first square is formed by 1 block; 1 is the first square. The next strip, made of white squares, shows how adding 3 more blocks makes another square: four. The next strip, made of black squares, shows how adding 5 more blocks makes the next square.

1. Explain how this process can be continued indefinitely.

2. Explain how the following diagram proves the same theorem: the sum of the first $n$ natural numbers equals $n^2$.

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1 Dunham, William (1974), *The Mathematical Universe*, John Wiley and Sons, p. 120
Mathematical folklore holds that the great **Carl Friedrich Gauss** (German mathematician and scientist; 1777 - 1855) was once, as a very young child, scolded by being sent to the coat closet with a slate to determine the sum of the first hundred numbers: $1 + 2 + 3 + ... + 99 + 100$.

The legend holds that he returned within a minute with the correct answer.

The figure below illustrates Gauss's method as it can be represented with blocks to determine the sum

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8.$$
7. Explain how the Figure below provides a proof without words which proves the general result in #6

Determining the sum $1 + 2 + 3 + \ldots (n - 2) + (n - 1) + n$

8. Determine the value of the following sums:
   
   $1 + 2 + 1 =
   1 + 2 + 3 + 2 + 1 =
   1 + 2 + 3 + 4 + 3 + 2 + 1 =
   1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 =

9. What pattern do you see? Describe this pattern using the language of an algebraic equation.

10. Create a proof without words for this result.

11. Determine the value of the following sums:

   $1 + 3
   1 + 3 + 5
   1 + 3 + 5 + 7
   1 + 3 + 5 + 7 + 9

12. What pattern do you see? Describe this pattern using the language of an algebraic equation.
13. Create a proof without words for this result.

14. Create proofs without words for the following infinite sums:
   a. \[ \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots = \frac{1}{2} \]
   b. \[ \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{1}{3} \]

15. Explain, in detail, the following proof without words.