4.3 Are You Divergent?(BC)

Series of Real Numbers

In the last lesson, you encountered several situations where you naturally considered an infinite sum of numbers called a geometric series. For example, by writing

$$N = 0.1212121\overline{212} = \left(\frac{12}{100}\right) + \left(\frac{12}{100}\right)\left(\frac{1}{100}\right) + \left(\frac{12}{100}\right)\left(\frac{1}{100}\right)^2 + \cdots$$

as a geometric series, you found a way to write the repeating decimal expansion of N as a single fraction: $N = \frac{4}{33}$. There are many other situations in mathematics where infinite sums of numbers arise, but often these are not geometric. In this section, you begin exploring these other types of infinite sums. Investigation 1 provides a context in which you see how one such sum is related to the famous number e.

Investigation 1: Have you ever wondered how your calculator can produce a numeric approximation for complicated numbers like e, π or ln (2)? After all, the only operations a calculator can really perform are addition, subtraction, multiplication, and division, the operations that make up polynomials. This investigation provides the first steps in understanding how this process works. Throughout the investigation, let $f(x) = e^x$.

a) Find the tangent line to f at $x = 0$ and use this linearization to approximate e. That is, find a formula $L(x)$ for the tangent line, and compute $L(1)$, since $L(1) \approx f(1) = e$

b) The linearization of e^x does not provide a good approximation to e since 1 is not very close to 0. To obtain a better approximation, alter your approach. Instead of using a straight line to approximate e, put an appropriate bend in your estimating function to make it better fit the graph of e^x for x close to 0. With the linearization, you had both $f(x)$ and $f'(x)$ share the same value as the linearization at $x = 0$. You will now use a quadratic approximation $P_2(x)$ to $f(x) = e^x$ centered at $x = 0$ which has the property that $P_2(0) = f(0)$, $P_2''(0) = f''(0)$ and $P_2'''(0) = f'''(0)$.

i. Let $P_2(x) = 1 + x + \frac{x^2}{2}$. Show that $P_2(0) = f(0)$, $P_2''(0) = f''(0)$ and

$P_2'''(0) = f'''(0)$. Then, use $P_2(x)$ to approximate e by observing that $P_2(1) \approx f(1)$.

ii. You can continue approximating e with polynomials of larger degree whose higher derivatives agree with those of f at 0. This turns out to make the polynomials fit the graph of f better for more values of x around 0. For example, let $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$. Show that $3_2(0) = f(0)$, $P_3''(0) = f''(0)$, $P_3'''(0) = f'''(0)$, and $P_3''''(0) = f''''(0)$. Use $P_3(x)$ to approximate e in a way similar to how you did so with $P_2(x)$ above.

Investigation 1 shows that an approximation to e using a linear polynomial is 2, an approximation to e using a quadratic polynomial is 2.5, and an approximation using a cubic polynomial is 2.6667. As you will see later, if you continue this process you can obtain approximations from quartic (degree 4), quintic (degree 5), and higher degree polynomials giving you the following approximations to e:

linear	$1+1$	2
quadratic	$1+1+\frac{1}{2}$	2.5
cubic	$1+1+\frac{1}{2}+\frac{1}{6}$	$2.\overline{6}$
quartic	$1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}$	$2.708\overline{3}$
quintic	$1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}$	$2.71\overline{6}$

You should see an interesting pattern here. The number e is being approximated by the sum

$$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots + \frac{1}{n!}.$$

for increasing values of n. And just as you did with Riemann sums, you can use the summation notation as a shorthand[1] for writing the sum, so that

$$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots + \frac{1}{n!} = \sum_{k=0}^{n} \frac{1}{k!}.$$

[1] Note that 0! appears in this equation. By definition, $0! = 1$.

You can calculate this sum using as large an n as you want, and the larger n is the more accurate the approximation this formula becomes. Ultimately, this argument shows that you can write the number e as the infinite sum

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

This sum is an example of a series (or an infinite series). Note that this series is the sum of the terms of the (infinite) sequence $\left\{\frac{1}{n!}\right\}$. In general, use the following notation and terminology.

Definition: An infinite series of real numbers is the sum of the entries in an infinite sequence of real numbers. In other words, an infinite series is sum of the form

$$a_1 + a_2 + \cdots a_n + \cdots = \sum_{k=1}^{\infty} a_k,$$

Where $a_1 + a_2$ ... are real numbers.

You will normally use summation notation to identify a series. If the series adds the entries of a sequence $\{a_n\}n \geq 1$, then you will write the series as

$$\sum_{k\geq 1} a_k,$$

or

$$\sum_{k\geq 1}^{\infty} a_k,$$

Note: each of these notations is simply shorthand for the infinite sum $a_1 + a_2 + \cdots + a_n + \cdots$

Is it even possible to sum an infinite list of numbers? This question is one whose answer shouldn't come as a surprise. After all, you have used the definite integral to add up continuous (infinite) collections of numbers, so summing the entries of a sequence might be even easier. Moreover, you have already examined the special case of geometric series in the previous section. The next investigation provides some more insight into how you make sense of the process of summing an infinite list of numbers.

Investigation 2: Consider the series

$$\sum_{k\geq 1}^{\infty} \frac{1}{k^2}$$

While it is physically impossible to add an infinite collection of numbers, you can, of course, add any finite collection of them. In what follows, you investigate how understanding how to find the nth partial sum (that is, the sum of the first n terms) enables you to make sense of the infinite sum.

a) Sum the first two numbers in this series. That is, find a numeric value for

$$\sum_{k=1}^{2} \frac{1}{k^2}$$

b) Next, add the first three numbers in the series.

c) Continue adding terms in this series to complete the table below. Carry each sum to at least 8 decimal places.

$\sum_{k=1}^{1} \frac{1}{k^2} =$		$\sum_{k=1}^{6} \frac{1}{k^2} =$	
$\sum_{k=1}^{2} \frac{1}{k^2} =$		$\sum_{k=1}^{7} \frac{1}{k^2} =$	
$\sum_{k=1}^{3} \frac{1}{k^2} =$		$\sum_{k=1}^{8} \frac{1}{k^2} =$	
$\sum_{k=1}^{4} \frac{1}{k^2} =$		$\sum_{k=1}^{9} \frac{1}{k^2} =$	
$\sum_{k=1}^{5} \frac{1}{k^2} =$		$\sum_{k=1}^{10} \frac{1}{k^2} =$	

d) The sums in the table in (c) form a sequence whose nth term is $S_n = \sum_{k=1}^{n} \frac{1}{k^2}$

Based on your calculations in the table, do you think the sequence $\{S_n\}$ converges or diverges? Explain. How do you think this sequence $\{S_n\}$ is related to the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$?

Investigation 2 illustrates how you define the sum of an infinite series. You can add up the first n terms of the series to obtain a new sequence of numbers (called the ***sequence of partial sums***). If that sequence converges, the corresponding infinite series is said to converge, and you can say that you can find the sum of the series.

Definition: The nth partial sum of the series $\sum_{k=1}^{\infty} a_k$, is the finite sum $S_n = \sum_{k=1}^{n} a_k$.

In other words, the nth partial sum S_n of a series is the sum of the first n terms in the series, or

$$S_n = a_1 + a_2 + \cdots a_n + \cdots$$

Now, investigate the behavior of a given series by examining the sequence

$$S_1, S_2, \dots, S_n, \dots$$

of its partial sums. If the sequence of partial sums converges to some finite number, then you say that the corresponding series *converges*. Otherwise, you say the series diverges. From your work in Investigation 2, the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

appears to converge to some number near 1.54977.

Formalize the concept of convergence and divergence of an infinite series in the following definition.

Definition: The infinite series

$$\sum_{k=1}^{\infty} a_k \,,$$

converges (or is **convergent**) if the sequence $\{S_n\}$ of partial sums converges, where

$$S_n = \sum_{k=1}^{n} a_k \,.$$

If $\lim\limits_{n\to\infty} S_n = S$, then you call S the sum of the series $\sum_{k=1}^{\infty} a_k$,

That is,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = S$$

If the sequence of partial sums does not converge, then the series **diverges** (or is **divergent**).

The early terms in a series do not contribute to whether or not the series converges or diverges. Rather, the convergence or divergence of a series

$$\sum_{k=1}^{\infty} a_k$$

is determined by what happens to the terms a_k for very large values of *k*. To see why, suppose that *m* is some constant larger than 1. Then

$$\sum_{k=1}^{\infty} a_k = (a_1 + a_2 + \cdots a_m) + \sum_{k=m+1}^{\infty} a_k$$

Since $a_1 + a_2 + \cdots a_m$ is a finite number, the series $\sum_{k=1}^{\infty} a_k$ a will converge if and only if the series $\sum_{k=m+1}^{\infty} a_k$ converges. Because the starting index of the series doesn't affect whether the series converges or diverges, you will often just write

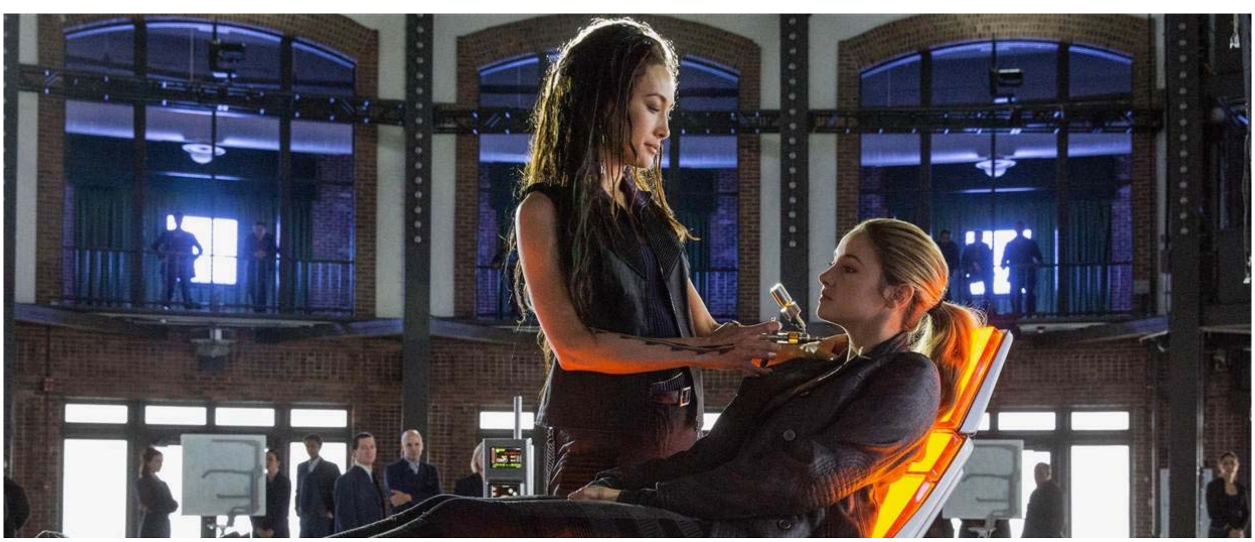
$$\sum a_k$$

when you are interested in questions of convergence/divergence and not necessarily the exact sum of a series.

In the last lesson, you encountered the special family of infinite geometric series whose convergence or divergence you determined. Recall that a geometric series is a special series of the form $\sum_{k=1}^{\infty} a_k$ where a and r are real numbers (and $r \neq 1$). You found that the nth partial sum Sn of a geometric series is given by the convenient formula

$$S_n = \frac{1 - r^n}{1 - r}$$

and thus a geometric series converges if $|r| < 1$. Geometric series diverge for all other values of *r*. While you have completely determined the convergence or divergence of geometric series, it is generally a difficult question to determine if a given nongeometric series converges or diverges. There are several tests you can use that you will consider in the following sections.



II. The Divergence Test

The first question you ask about any infinite series is usually "Does the series converge or diverge?" There is a straightforward way to check that certain series diverge; you explore this test in the next investigation.

Investigation 3: If the series $\sum a_k$ converges, then an important result necessarily follows regarding the sequence $\{a_n\}$. This investigation explores this result.

Assume that the series $\sum_{k=1}^{\infty} a_k$ converges and has sum equal to L.

a) What is the nth partial sum S_n of the series $\sum_{k=1}^{\infty} a_k$?

b) What is the (n - 1)st partial sum S_{n-1} of the series $\sum_{k=1}^{\infty} a_k$?

c) What is the difference between the nth partial sum and the (n - 1)st partial sum of the series $\sum_{k=1}^{\infty} a_k$?

d) Since you are assuming that $\sum_{k=1}^{\infty} a_k = L$, what does that tell you about $\lim_{n\to\infty} S_n$? Why? What does that tell you about $\lim_{n\to\infty} S_{n-1}$? Why?

e) Combine the results of the previous two parts of this investigation to determine $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1})$.

The result of Investigation 3 is the following important conditional statement:

> If the series $\sum_{k=1}^{\infty} a_k$ converges, then the sequence $\{a_k\}$ of k^{th} terms converges to 0.

It is logically equivalent to say that if the sequence a of n terms does not converge to 0, then the series $\sum_{k=1}^{\infty} a_k$ cannot converge. This statement is called the Divergence Test.

> **The Divergence Test**: If $\lim_{k\to\infty} a_k \neq 0$, then the series $\sum a_k$, diverges.

Investigation 4: Determine if the Divergence Test applies to the following series. If the test does not apply, explain why. If the test does apply, what does it tell you about the series?

a) $\sum \frac{k}{k+1}$

b) $\sum (-1)^k$

c) $\sum \frac{1}{k}$

Note: be very careful with the Divergence Test. This test only tells you what happens to a series if the terms of the corresponding sequence do not converge to 0. If the sequence of the terms of the series does converge to 0, the Divergence Test does not apply: indeed, as you will soon see, a series whose terms go to zero may either converge or diverge.

III. The Integral Test

The Divergence Test settles the questions of divergence or convergence of series $\sum a_k$ in which $\lim_{k\to\infty} a_k \neq 0$. Determining the convergence or divergence of series $\sum a_k$ in which $\lim_{k\to\infty} a_k = 0$ turns out to be more complicated. Often, you have to investigate the sequence of partial sums or apply some other technique.

As an example, consider the harmonic series[2]

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

The table below shows some partial sums of this series.

$\sum_{k=1}^{1} \frac{1}{k}$	1	$\sum_{k=1}^{6} \frac{1}{k}$	2.450000000
$\sum_{k=1}^{2} \frac{1}{k}$	1.5	$\sum_{k=1}^{7} \frac{1}{k}$	2.592857143
$\sum_{k=1}^{3} \frac{1}{k}$	1.833333333	$\sum_{k=1}^{8} \frac{1}{k}$	2.717857143
$\sum_{k=1}^{4} \frac{1}{k}$	2.083333333	$\sum_{k=1}^{9} \frac{1}{k}$	2.828968254
$\sum_{k=1}^{5} \frac{1}{k}$	2.283333333	$\sum_{k=1}^{10} \frac{1}{k}$	2.928968254

This information doesn't seem to be enough to tell you if the series $\sum_{k=1}^{\infty} \frac{1}{k}$ converges or diverges. The partial sums could eventually level off to some fixed number or continue to grow without bound. Even if you look at larger partial sums, such as $\sum_{k=1}^{1000} \frac{1}{k} \approx 7.485470861$, the result doesn't particularly sway you one way or another.

The Integral Test is one way to determine whether or not the harmonic series converges, and you explore this further in the next investigation.

[2] This series is called harmonic because each term in the series after the first is the harmonic mean of the term before it and the term after it. The harmonic mean of two numbers a and b is $\frac{2ab}{a+b}$. See "What's Harmonic about the Harmonic Series", by David E. Kullman (in the College Mathematics Journal, Vol. 32, No. 3 (May, 2001), 201-203) for an interesting discussion of the harmonic mean.

Investigation 5: Consider the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. Recall that the harmonic series will converge if its sequence of partial sums converges.

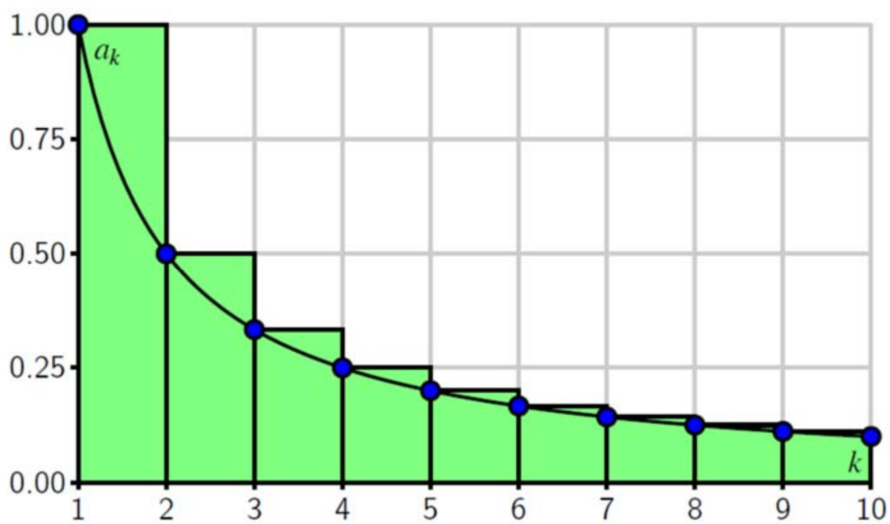
The n^th partial sum S_n of the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is

$$S_n = \sum_{k=1}^{\infty} \frac{1}{k}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$= 1(1) + (1)\frac{1}{2} + (1)\frac{1}{3} + \cdots + (1)\frac{1}{n}$$

Through this last expression for S_n, you can visualize this partial sum as a sum of areas of rectangles with heights 1 and bases of length 1, as shown below, which uses the 9th partial sum.



The graph of the continuous function f defined by $f(x) = \frac{1}{x}$ is overlaid on this plot.

a) Explain how this picture represents a particular Riemann sum.

b) What is the definite integral that corresponds to the Riemann sum you considered in (a)?

c) Which is larger, the definite integral in (b), or the corresponding partial sum S_n of the series? Why?

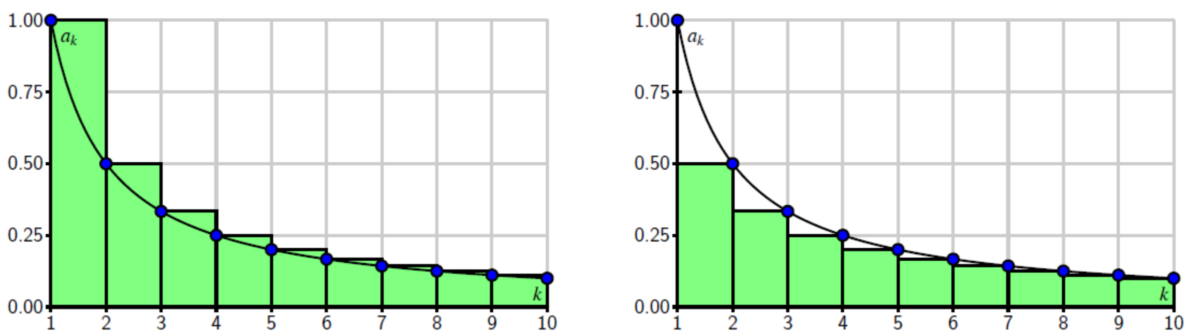
d) If instead of considering the 9^th partial sum, you consider the n^{th} partial sum, and you let n go to infinity, you can then compare the series $\sum_{k=1}^{\infty} \frac{1}{k}$ to the improper integral $\int_1^{\infty} \frac{1}{x}\, dx$. Which of these quantities is larger? Why?

e) Does the improper integral $\int_1^{\infty} \frac{1}{x}$ converge or diverge? What does that result, together with your work in (d), tell you about the series $\sum_{k=1}^{\infty} \frac{1}{k}$?

The ideas from Investigation 5 hold more generally. Suppose that *f* is a continuous decreasing function and that $a_k = f(x)$for each value of *k*. Consider the corresponding series $\sum_{k=1}^{\infty} \frac{1}{k}$. The partial sum

$$S_n = \sum_{k=1}^{\infty} a_k$$

can always be viewed as a left-hand Riemann sum of $f(x)$ using rectangles with heights given by the values a_k and bases of length 1. A representative picture is shown below left.



Since *f* is a decreasing function, you have that

$$S_n > \int_1^n f(x)dx$$

Taking limits as *n* goes to infinity shows that

$$\sum_{k=1}^{\infty} a_k > \int_1^{\infty} f(x)dx$$

Therefore, if the improper integral $\int_1^{\infty} f(x)dx$ diverges, so does the series $\sum_{k=1}^{\infty} \frac{1}{k}$.

Wat's more, if you look at the right-hand Riemann sums of *f* on [1, n] as shown above right, you see that

$$\int_1^\infty f(x)dx > \sum_{k=2}^{\infty} a_k$$

So if $\int_1^\infty f(x)dx$ converges, then so does $\sum_{k=2}^\infty \frac{1}{k}$, which also means that the series

$\sum_{k=1}^\infty \frac{1}{k}$ converges. The preceding discussion demonstrates the truth of the Integral Test.

The Integral Test: Let *f* be a real valued function and assume *f* is decreasing and positive for all *x* larger than some number *c*.
Let $a_k = f(x)$ for each positive integer *k*.

1. If the improper integral $\int_c^\infty f(x)dx$ converges, then the series $\sum_{k=1}^\infty a_k$ converges.
2. If the improper integral $\int_c^\infty f(x)dx$ diverges, then the series $\sum_{k=1}^\infty a_k$ diverges.

Note that the Integral Test compares a given infinite series to a natural, corresponding improper integral and basically says that the infinite series and corresponding improper integral both have the same convergence status. In the next investigation, you will apply the Integral Test to determine the convergence or divergence of a class of important series.

Investigation 6: The series $\sum_{k=1}^\infty \frac{1}{k^p}$ are special series called p-series. You have already seen that the p-series with $p = 1$ (the harmonic series) diverges. You investigate the behavior of other p-series in this investigation.

a) Evaluate the improper integral $\int_1^\infty \frac{1}{x^2}dx$. Does the series $\sum_{k=1}^\infty \frac{1}{k^2}$ converge or diverge? Explain.

b) Evaluate the improper integral $\int_1^\infty \frac{1}{x^p}\,dx$ where $p > 1$. For which values of p can you conclude that the series $\sum_{k=1}^\infty \frac{1}{k^p}$ converges?

c) Evaluate the improper integral $\int_1^\infty \frac{1}{x^p}\,dx$ where $p < 1$. What does this tell you about the corresponding p-series $\sum_{k=1}^\infty \frac{1}{k^p}$?

d) Summarize your work in this investigation by completing the following statement.

The p-series $\sum_{k=1}^\infty \frac{1}{k^p}$ converges if and only if ________________________.

IV. The Limit Comparison Test

The Integral Test allows you to determine the convergence of an entire family of series: the p-series. However, you have seen that it is, in general, difficult to integrate functions, so the Integral Test is not one that you can use all of the time. In fact, even for a relatively simple series like $\sum \frac{k^2+1}{k^4+2k+2}$, the Integral Test is not an option. In this section you will develop a test that you can use to apply to series of rational functions like this by comparing their behavior to the behavior of p-series.

Investigation 7: Consider the series $\sum \frac{k+1}{k^3+2}$. Since the convergence or divergence of a series only depends on the behavior of the series for large values of k, you might examine the terms of this series more closely as k gets large.

a) By computing the value of $\frac{k+1}{k^3+2}$ for $k = 100$ and $k = 1000$, explain why the terms $\frac{k+1}{k^3+2}$ are essentially $\frac{k}{k^3}$ when k is large.

b) Formalize your observations in (a) a bit more. Let $a_k = \frac{k+1}{k^3+2}$ and $b_k = \frac{k}{k^3}$. Calculate $\lim_{k\to\infty} \frac{a_k}{b_k}$. What does the value of the limit tell you about a_k and b_k for large values of k? Compare your response from part (a).

c) Does the series $\sum \frac{k}{k^3}$ converge or diverge? Why? What do you think that tells you about the convergence or divergence of the series $\sum \frac{k+1}{k^3+2}$? Explain.

Investigation 7 illustrates how you can compare one series with positive terms to another whose behavior (that is, whether the series converges or diverges) you know. More generally, suppose you have two series Σa_k and Σb_k with positive terms and you know the behavior of the series Σa_k. Recall that the convergence or divergence of a series depends only on what happens to the terms of the series for large values of k, so if you know that a_k and b_k are essentially proportional to each other for large k, then the two series Σa_k and Σb_k should behave the same way. In other words, if there is a positive finite constant c such that

$$\lim_{k \to \infty} \frac{a_k}{b_k} = c,$$

then $b_k \approx c a_k$ for large values of k. So

$$\Sigma b_k \approx \Sigma c a_k = c \Sigma a_k.$$

Since multiplying by a nonzero constant does not affect the convergence or divergence of a series, it follows that the series Σa_k and Σb_k either both converge or both diverge. The formal statement of this fact is called the Limit Comparison Test.

The Limit Comparison Test: Let Σa_k and Σb_k be series with positive terms. If

$$\lim_{k \to \infty} \frac{a_k}{b_k} = c,$$

For some positive (finite) term, then Σa_k and Σb_k either converge or diverge.

In essence, the Limit Comparison Test shows that if you have a series $\sum \frac{p(k)}{q(k)}$ of rational functions where $p(k)$ is a polynomial of degree m and $q(k)$ a polynomial of degree l, then the series $\sum \frac{p(k)}{q(k)}$ will behave like the series $\sum \frac{k^m}{k^t}$. So this test allows you to quickly and easily determine the convergence or divergence of series whose summands are rational functions.

Investigation 8: Use the Limit Comparison Test to determine the convergence or divergence of the series

$$\sum \frac{3k^2+1}{5k^4+2k+2}.$$

by comparing it to the series $\sum \frac{1}{k^2}$.

V. The Ratio Test

The Limit Comparison Test works well if you can find a series with known behavior to compare. But such series are not always easy to find. In this section you will examine a test that allows you to examine the behavior of a series by comparing it to a geometric series, without knowing in advance which geometric series you need.

Investigation 9:

Consider the series defined by

$$\sum_{k=1}^{\infty} \frac{2^k}{3^k - k}$$

This series is not a geometric series, but this investigation will illustrate how you might compare this series to a geometric one. Recall that a series a_k is geometric if the ratio $\frac{a_k+1}{a_k}$ is always the same. For the series above, note that $a_k = \frac{2^k}{3^k-k}$k

a) To see if $\sum \frac{2^k}{3^k-k}$ is comparable to a geometric series, you analyze the ratios of successive terms in the series. Complete the Table below, listing your calculations to at least 8 decimal places.

k	$\frac{a_k + 1}{a_k}$
5	
10	
20	
21	
22	
23	
24	
25	

b) Based on your calculations in the Table, what can you say about the ratio $\frac{a_k+1}{a_k}$ if k is large?

c) Do you agree or disagree with the statement: "the series $\sum \frac{2^k}{3^k - k}$ is approximately geometric when k is large"? If not, why not? If so, do you think the series $\sum \frac{2^k}{3^k - k}$ converges or diverges? Explain.

The Ratio Test: Let $\sum a_k$ be an infinite series. Suppose

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$$

1. If $0 \leq r < 1$, then the series $\sum a_k$ converges.

2. If $1 < \mathrm{r}$, then the series $\sum a_k$ diverges.

3. If $\mathrm{r} = 1$, then the test is inconclusive.

Note well: The Ratio Test takes a given series and looks at the limit of the ratio of consecutive terms; in so doing, the test is essentially asking, "is this series approximately geometric?" If the series can be thought of as essentially geometric, the test use the limiting common ratio to determine if the given series converges.

You have now encountered several tests for determining convergence or divergence of series. The Divergence Test can be used to show that a series diverges, but never to prove that a series converges. You used the Integral Test to determine the convergence status of an entire class of series, the p-series. The Limit Comparison Test works well for series that involve rational functions and which can therefore by compared to p-series. Finally, the Ratio Test allows you to compare your series to a geometric series; it is particularly useful for series that involve nth powers and factorials. Two other tests, the Direct Comparison Test and the Root Test, are discussed in the lesson. Now it is time for some practice.

VI. Exercises

1. Determine whether each of the following series converges or diverges. Explicitly state which test you use.

a) $\sum \frac{k}{2^k}$

b) $\sum \frac{k^3+2}{k^2+1}$

c) $\sum \frac{10^k}{k!}$

d) $\sum \frac{k^3+2k^2+1}{k^6+4}$

2. In this exercise you investigate the sequence $\left\{\frac{b^n}{n!}\right\}$ for any constant b.

a) Use the Ratio Test to determine if the series $\sum \frac{10^k}{k!}$ converges or diverges.

b) Now apply the Ratio Test to determine if the series $\sum \frac{b^k}{k!}$ converges for any constant b.

c) Use your result from (b) to decide whether the sequence of $\left\{\frac{b^n}{n!}\right\}$ converges or diverges. If the sequence $\left\{\frac{b^n}{n!}\right\}$ converges, to what does it converge? Explain your reasoning.

3. There is a test for convergence similar to the Ratio Test called the Root Test. Suppose you have a series $\sum a_k$ of positive terms so that $a_n \to 0$ as $a_n \to 0$.

a) Assume

$$\sqrt[n]{a_n} \to r$$

as *n* goes to infinity. Explain why this tells you that $a_n \approx r^n$ for large values of *n*.

b) Using the result of part (a), explain why $\sum a_k$ looks like a geometric series when *n* is big. What is the ratio of the geometric series to which $\sum a_k$ is comparable?

c) Use what you know about geometric series to determine that values of *r* so that $\sum a_k$ converges if $\sqrt[n]{a_n} \to r$ as $n \to \infty$.

4. The associative and distributive laws of addition allow you to add finite sums in any order you want. That is, if $\sum_{k=0}^{n} a_k$ and $\sum_{k=0}^{n} b_k$ are finite sums of real numbers, then

$$\sum_{k=0}^{n} a_k + \sum_{k=0}^{n} b_k = \sum_{k=0}^{n}(a_k + b_k).$$

However, you do need to be careful extending rules like this to infinite series.

a) Let $a_n = 1 + \frac{1}{2^n}$ and $b_n = -1$ for each nonnegative integer *n*.

(i) Explain why the series $\sum_{k=0}^{n} a_k$ and $\sum_{k=0}^{n} b_k$ both diverge.

(ii) Explain why the series $\sum_{k=0}^{n}(a_{k+}b_k)$ converges.

(iii) Explain why

$$\sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k \neq \sum_{k=0}^{\infty}(a_k + b_k).$$

This shows that it is possible to have two divergent series $\sum_{k=0}^{n} a_k$ and $\sum_{k=0}^{n} b_k$ but yet have the series $\sum_{k=0}^{n}(a_{k+}b_k)$ converge.

b) While part (a) shows that you cannot add series term by term in general, you can under reasonable conditions. The problem in part (a) is that you tried to add divergent series. In this exercise you will show that if a and b are convergent series, then $\sum (a_{k+}b_k)$ is a convergent series and

$$\sum(a_k + b_k) = \sum a_k + \sum b_k.$$

(i) Let A_n and B_n be the n^{th} partial sums of the series $\sum_{k=1}^{n} a_k$ and $\sum_{k=1}^{n} b_k$, respectively. Explain why

$$A_n + B_n = \sum_{k=1}^{n}(a_k + b_k).$$

(ii) Use the previous result and properties of limits to show that

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

(Note that the starting point of the sum is irrelevant in this problem, so it doesn't matter where you begin the sum.)

c) Use the prior result to find the sum of the series $\sum_{k=0}^{\infty} \frac{2^k+3^k}{5^k}$

5. In the Limit Comparison Test you compared the behavior of a series to one whose behavior you know. In that test you use the limit of the ratio of corresponding terms of the series to determine if the comparison is valid. In this exercise you see how you can compare two series directly, term by term, without using a limit of sequence. First you consider an example.

(a) Consider the series

$$\sum \frac{1}{k^2} \quad \text{and} \quad \sum \frac{1}{k^2 + k}$$

You know that the series $\sum \frac{1}{k^2}$ is a p-series with $p = 2 > 1$ and so $\sum \frac{1}{k^2}$

converges. In this part of the exercise you will see how to use information about $\sum \frac{1}{k^2}$ to determine information about $\sum \frac{1}{k^2+k}$. Let $a_k = \frac{1}{k^2}$ and $b_k = \frac{1}{k^2+k}$.

(i) Let S_n be the n[th] partial sum of $\sum \frac{1}{k^2}$ and T_n be the n[th] partial sum of $\sum \frac{1}{k^2+k}$.

Which is larger, S_1 or T_1? Why?

(ii) Recall that

$$S_2 = S_1 + a_2 \quad \text{and} \quad T_2 = T_1 + b_2$$

Which is larger, a_2 or b_2? Based on that answer, which is larger, S_2 or T_2?

(iii) Recall that

$$S_3 = S_2 + a_3 \quad \text{and} \quad T_3 = T_2 + b_3$$

Which is larger, a_3 or b_3? Based on that answer, which is larger, S_3 or T_3?

(iv) Which is larger, a_n or b_n? Explain. Based on that answer, which is larger, S_n or T_n?

(v) Based on your response to the previous part of this exercise, what relation- ship do you expect there to be between $\sum \frac{1}{k^2}$ and $\sum \frac{1}{k^2+k}$? Do you expect $\sum \frac{1}{k^2+k}$ to converge or diverge? Why?

b) The example in the previous part of this exercise illustrates a more general result. Explain why the Direct Comparison Test, stated here, works.

The Direct Comparison Test: If

$$0 \leq b_k \leq a_k$$

For every k, then

$$0 \leq \sum b_k \leq \sum a_k$$

1. If $\sum a_k$ converges, then $\sum b_k$ converges.

2. If $\sum a_k$ diverges, then $\sum b_k$ diverges.

VII. Formative Assessment – Khan Academy

Complete the following online practice exercises in the *Sequences and Series* unit of Khan Academy's AP Calculus BC course:

- https://www.khanacademy.org/math/ap-calculus-bc/series-bc/infinite-geo-series-bc/e/geometric-series-of-constants