4.2 Therapeutic Concentration Levels (BC)

Introduction to Series

Many important sequences are generated through the process of addition. In Investigation 1, you see a particular example of a special type of sequence that is connected to a sum.

Investigation 1: Warfarin is an anticoagulant that prevents blood clotting; often it is prescribed to stroke victims in order to help ensure blood flow. The level of Warfarin has to reach a certain concentration in the blood in order to be effective.

Suppose Warfarin is taken by a particular patient in a 5 mg dose each day. The drug is absorbed by the body and some is excreted from the system between doses. Assume that at the end of a 24-hour period, 8% of the drug remains in the body. Let $Q(n)$ be the amount (in mg) of Warfarin in the body before the $(n+1)^{st}$ dose of the drug is administered.

1a) Explain why $Q(1) = 5 \times 0.08$ mg.

b) Explain why $Q(2) = \big(5 + Q(1)\big) \cdot 0.08$ mg. Then show that
$Q(2) = (5 \times 0.08)\ (1 + 0.08)$ mg.

c) Explain why $Q(3) = \big(5 + Q(2)\big) \cdot 0.08$. Then show that

$Q(3) = (5 \times 0.08)\ 1 + 0.08 + 0.082$ mg.

d) Explain why $Q(4) = \big(5 + Q(3)\big) \cdot 0.08$ mg. Then show that

$Q(4) = (5 \times 0.08)\ 1 + 0.08 + 0.082 + 0.083$ mg.

e) There is a pattern that you should see emerging. Use this pattern to find a formula for $Q(n)$, where n is an arbitrary positive integer.

f) Complete the table below with values of Q n for the provided n-values (reporting $Q(n)$ to 10 decimal places). What appears to be happening to the sequence $Q(n)$ as n increases?

$Q(1)$	0.40
$Q(2)$	
$Q(3)$	
$Q(4)$	
$Q(5)$	
$Q(6)$	
$Q(7)$	
$Q(8)$	
$Q(9)$	
$Q(10)$	

II. Finite Geometric Series

In Investigation 1 you encountered the sum

$$(5 \times 0.08)\ (1 + 0.08 + 0.08^2 + 0.08^3 + \cdot\cdot\cdot + 0.08^{n-1}).$$

In order to evaluate the long-term level of Warfarin in the patient's system, you will need to fully understand the sum in this expression. This sum has the form

$$a + ar + ar^2 + \cdots + r^{n-1}$$

where $a = 5\ \ 0.08$ and $r = 0.08$. Such a sum is called a ***geometric sum*** with ratio r. You will analyze this sum in more detail in the next Investigation.

Investigation 2: Let a and r be real numbers (with $r \neq 1$) and let

$$S_n = a + ar + ar^2 + \cdots + r^{n-1}$$

Tour goal is to find a shortcut formula for S_n that does not involve a sum of n

terms.

2a) Multiply S_n by r. What does the resulting sum look like?

b) Subtract rS_n from S_n and explain why

$$S_n - rS_n = a + r^{n-1}$$

c) Solve the equation in (b) for S_n to find a simple formula for S_n that does not involve adding n terms.

You can summarize the result of Investigation 2 in the following way.

A geometric sum S_n is a sum of the form

$$S_n = a + ar + ar^2 + \cdots + r^{n-1}$$

where a and r are real numbers such that $r \neq 1$. The geometric sum S_n can be written more simply as

$$S_n - rS_n = a + r^{n-1}$$

This can be further simplified as

$$S_n = a\frac{1 - r^n}{1 - r}$$

Investigation 3:

3a) Calculate the following finite geometric series, using the simplified formula from Investigation 2.

a) $4 + 12 + 36 + 108 + + 324 + 972 + 2916 + 8748$

b) $\sum_{n=1}^{50} 7(4^k)$

c) find the sum of the first 125 terms of the following: $1024 + 512 + 256 + \ldots$

d) find the sum of the first 53 terms of the following: $\frac{1}{2} + \frac{3}{4} + \frac{9}{8} + \cdots$

e) find the sum of the first 53 terms of the following: $13 + (-26) + 52 + (-104) + \cdots$

f) find the sum of the first 125 terms of the following recursively defined sequence, without a calculator.

$a_i = 6 \cdot a_{i-1}, \;\; a_1 = 15, \quad \text{for all } i \geq 2$

III. Formative Assessment – Khan Academy

Complete the following online practice exercises in the *Sequences and Series* unit of Khan Academy's AP Calculus BC course:

- https://www.khanacademy.org/math/ap-calculus-bc/series-bc/geo-series-bc/e/geometric-series--1

IV. Partial Sums

Returning to the example involving Warfarin from Investigation 1, the real question that one is interested in is what is the long-term effect of the drug in a patient's bloodstream?

Apply the simplified equation to the example, and notice that

$$Q(n) = (5 \times 0.08)\ (1 + 0.08 + 0.08^2 + 0.08^3 + \cdots + 0.08^{n-1})\text{ mg},$$

so $Q(n)$ is a geometric sum with $a = 5 \times 0.08 = 0.4$ and $r = 0.08$. Thus,

$$Q(n) = 0.4\left(\frac{1 - 0.08^n}{1 - 0.08}\right) = \frac{1}{2.3}(1 - 0.08^n)$$

Notice that as n goes to infinity, the value of $0.08n$ goes to 0. So,

$$\lim_{n\to\infty} Q(n) = \lim_{n\to\infty} \frac{1}{2.3}(1 - 0.08^n) = \frac{1}{2.3} \approx 0.435$$

Therefore, the long-term level of Warfarin in the blood under these conditions is $\frac{1}{2.3}$, which is approximately 0.435 mg.

To determine the long-term effect of Warfarin, you considered a geometric sum of n terms, and then considered what happened as n was allowed to grow without bound. In this sense, you were actually interested in an infinite geometric sum (the result of letting n go to infinity in the finite sum). You call such an infinite geometric sum a ***geometric series***.

Definition 4.2. A geometric series is an infinite sum of the form

$$a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n$$

The value of r in the geometric series is called the ***common ratio*** of the series because the ratio of the $(n+1)$st term ar^n to the nth term ar^{n-1} is always r.

Geometric series are very common in mathematics and arise naturally in many different situations. As a familiar example, suppose you want to write the number with repeating decimal expansion

$$N = 0.121212\overline{12}$$

as a rational number. Observe that

$$N = 0.121212\overline{12}$$

$$= \left(\frac{12}{100}\right) + \left(\frac{12}{100}\right)\left(\frac{1}{100}\right) + \left(\frac{12}{100}\right)\left(\frac{1}{100}\right)^2 + \cdots$$

which is an infinite geometric series with a $= \frac{12}{100}$ and $r = \frac{1}{100}$. In the same way that you were able to find a shortcut formula for the value of a (finite) geometric sum, you would like to develop a formula for the value of a (infinite) geometric series. You explore this idea in the following investigation.

Investigation 4: Let $r \neq 1$ and a be real numbers and let

$$S = a + ar + ar^2 + \cdots + r^{n-1} \ldots$$

be an infinite geometric series. For each positive integer n, let

$$S_n = a + ar + ar^2 + \cdots + r^{n-1}.$$

Recall that

$$S_n = a\frac{1 - r^n}{1 - r}$$

a) What should you allow n to approach in order to have S_n approach S?

b) What is the value of $\lim_{n\to\infty} r^n$ for

- $|r\,| > 1$?
- $|r\,| < 1$?

Explain.

c) If $|r\,| < 1$, use the formula for S_n and your observations in (a) and (b) to explain why S is finite and find a resulting formula for S.

From your work in Investigation 3, you can now find the value of the geometric series $N = \left(\frac{12}{100}\right) + \left(\frac{12}{100}\right)\left(\frac{1}{100}\right) + \left(\frac{12}{100}\right)\left(\frac{1}{100}\right)^2 + \cdots$. In particular, using $a = \frac{12}{100}$ and $r = \frac{1}{100}$, you see that

$$N = \frac{12}{100}\left(\frac{1}{1-\frac{1}{100}}\right) = \frac{12}{100}\left(\frac{100}{99}\right) = \frac{4}{33}.$$

It is important to notice that a geometric sum is simply the sum of a finite number of terms of a geometric series. In other words, the geometric sum S_n for the geometric series

$$\sum_{n=0}^{\infty} ar^k$$

is

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{n=0}^{\infty} ar^n$$

You also call this sum S_n the nth ***partial sum*** of the geometric series. Summarize your recent work with geometric series as follows.

A geometric series is an infinite sum of the form

$$a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n$$

where a and r are real numbers such that $r \neq 1$.

- The nth partial sum S_n of the geometric series is

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

- If $|r| < 1$, then using the fact that $S_n = a\frac{1-r^n}{1-r}$, it follows that the sum S of the geometric series is

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} a\frac{1-r^n}{1-r} = \frac{a}{1-r}$$

A *partial sum* of an infinite series is the sum of a finite number of consecutive terms beginning with the first term. When working with infinite series, it is often helpful to examine the behavior of the partial sums.

Investigation 5: Compute the limit of the partial sums to determine whether the series converges or diverges:

5a) $1 + 1.1 + 1.11 + 1.111 + 1.1111 + \ldots$

b) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} +$

c) $3 + 0.5 + 0.05 + 0.005 + 0.0005 + \ldots$

Investigation 6: The formulas you have derived for the geometric series and its partial sum so far have assumed you begin indexing your sums at $n = 0$. If instead you have a sum that does not begin at $n = 0$, you can factor out common terms and use your established formulas. This process is illustrated in the examples in this activity.

6a) Consider the sum

$$\sum_{n=1}^{\infty}(2)\left(\frac{1}{3}\right)^{k} = (2)\left(\frac{1}{3}\right) + (2)\left(\frac{1}{3}\right)^{2} + (2)\left(\frac{1}{3}\right)^{3} + \cdots$$

Remove the common factor of $(2)\left(\frac{1}{3}\right)$ from each term and hence find the sum of the series.

b) Next let a and r be real numbers with $-1 < r < 1$. Consider the sum

$$\sum_{n=3}^{\infty} ar^{k} = ar^{3} + ar^{4} + ar^{5} \ldots$$

Remove the common factor of ar^3 from each term and find the sum of the series.

c) Finally, you consider the most general case. Let a and r be real numbers with

$-1 < \mathrm{r} < 1$, let n be a positive integer, and consider the sum

$$\sum_{n=n}^{\infty} ar^{k} = ar^{n} + ar^{n+1} + ar^{n+2} \ldots$$

Remove the common factor of ar^n from each term to find the sum of the series.

VI. Exercises

1. There is an old question that is often used to introduce the power of geometric growth. Here is one version. Suppose you are hired for a one month (30 days, working every day) job and are given two options to be paid.

Option 1. You can be paid $500 per day or

Option 2. You can be paid 1 cent the first day, 2 cents the second day, 4 cents the third day, 8 cents the fourth day, and so on, doubling the amount you are paid each day.

a) How much will you be paid for the job in total under Option 1?

b) Complete the table below to determine the pay you will receive under Option 2 for the first 10 days.

Day	Pay on this day	Total amount paid to date
1	$ 0.01	$ 0.01
2	$ 0.02	$ 0.03
3		
4		
5		
6		
7		
8		
9		
10		

c) Find a formula for the amount paid on day *n*, as well as for the total amount paid by day *n*. Use this formula to determine which option (1 or 2) you should take.

2. Suppose you drop a golf ball onto a hard surface from a height h. The collision with the ground causes the ball to lose energy and so it will not bounce back to its original height. The ball will then fall again to the ground, bounce back up, and continue. Assume that at each bounce the ball rises back to a height $\frac{3}{4}$ of the height from which it dropped. Let h_n be the height of the ball on the nth bounce, with $h_0 = h$. In this exercise you will determine the distance traveled by the ball and the time it takes to travel that distance.

a) Determine a formula for h_1 in terms of h.

b) Determine a formula for h_2 in terms of h.

c) Determine a formula for h_3 in terms of h.

d) Determine a formula for h_n in terms of h.

e) Write an infinite series that represents the total distance traveled by the ball. Then determine the sum of this series.

f) Next, let's determine the total amount of time the ball is in the air.

i) When the ball is dropped from a height H, if you assume the only force acting on it is the acceleration due to gravity, then the height of the ball at time t is given by

$$H - \frac{1}{2}gt^2$$

Use this formula to determine the time it takes for the ball to hit the ground after being dropped from height H.

ii) Use your work in the preceding item, along with that in (a)-(e) above to determine the total amount of time the ball is in the air.

3. Suppose you play a game with a friend that involves rolling a standard six-sided die. Before a player can participate in the game, he or she must roll a six with the die. Assume that you roll first and that you and your friend take alternate rolls. In this exercise you will determine the probability that you roll the first six.

a) Explain why the probability of rolling a six on any single roll (including your first turn) is $\frac{1}{6}$.

b) If you don't roll a six on your first turn, then in order for you to roll the first six on your second turn, both you and your friend had to fail to roll a six on your first turns, and then you had to succeed in rolling a six on your second turn. Explain why the probability of this event is

$$\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)=\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right).$$

c) Now suppose you fail to roll the first six on your second turn. Explain why the probability is

$$\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)=\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right),$$

that you to roll the first six on your third turn.

d) The probability of you rolling the first six is the probability that you roll the first six on your first turn plus the probability that you roll the first six on your second turn plus the probability that you roll the first six on your third turn, and so on. Explain why this probability is

$$\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)+\cdots\ .$$

Find the sum of this series and determine the probability that you roll the first six.

4. The goal of a federal government stimulus package is to positively affect the economy. Economists and politicians quote numbers like "k million jobs and a net stimulus to the economy of n billion of dollars." Where do they get these numbers? Let's consider one aspect of a stimulus package: tax cuts. Economists understand that tax cuts or rebates can result in long-term spending that is many times the amount of the rebate. For example, assume that for a typical person, 75% of her entire income is spent (that is, put back into the economy). Further, assume the government provides a tax cut or rebate that totals P dollars for each person.

a) The tax cut of P dollars is income for its recipient. How much of this tax cut will be spent?

b) In this simple model, the spent portion of the tax cut/rebate from part (a) then becomes income for another person who, in turn, spends 75% of this income. After this "second round" of spent income, how many total dollars have been added to the economy as a result of the original tax cut/rebate?

c) This second round of spending becomes income for another group who spend

75% of this income, and so on. In economics this is called the multiplier effect. Explain why an original tax cut/rebate of P dollars will result in multiplied spending of

$$0.75P(1 + 0.75 + 0.75^2 + \cdots).$$

dollars.

d) Based on these assumptions, how much stimulus will a 200 billion dollar tax cut/rebate to consumers add to the economy, assuming consumer spending remains consistent forever.

5. Like stimulus packages, home mortgages and foreclosures also impact the economy. A problem for many borrowers is the adjustable rate mortgage, in which the interest rate can change (and usually increases) over the duration of the loan, causing the monthly payments to increase beyond the ability of the borrower to pay. Most financial analysts recommend fixed rate loans, ones for which the monthly payments remain constant throughout the term of the loan. In this exercise you will analyze fixed rate loans.

When most people buy a large ticket item like car or a house, they have to take out a loan to make the purchase. The loan is paid back in monthly installments until the entire amount of the loan, plus interest, is paid. With a loan, you borrow money, say P dollars (called the principal), and pay off the loan at an interest rate of r%. To pay back the loan you make regular monthly payments, some of which goes to pay off the principal and some of which is charged as interest. In most cases, the interest is computed based on the amount of principal that remains at the beginning of the month. You assume a fixed rate loan, that is one in which you make a constant monthly payment M on your loan, beginning in the original month of the loan.

Suppose you want to buy a house. You have a certain amount of money saved to make a down payment, and you will borrow the rest to pay for the house. Of course, for the privilege of loaning you the money, the bank will charge you interest on this loan, so the amount you pay back to the bank is more than the amount you borrow. In fact, the amount you ultimately pay depends on three things: the amount you borrow (called the *principal*), the interest rate, and the length of time you have to pay off the loan plus interest (called the *duration* of the loan). For this example, you assume that the interest rate is fixed at r%.

To pay off the loan, each month you make a payment of the same amount (called *installments*). Suppose you borrow P dollars (your principal) and pay off the loan at an interest rate of r% with regular monthly installment payments of M dollars. So in month 1 of the loan, before you make any payments, your principal is P dollars. Your goal in this exercise is to find a formula that relates these three parameters to the time duration of the loan.

You are charged interest every month at an annual rate of r%, so each month you pay $\frac{r}{12}$% interest on the principal that remains. Given that the original principal is P dollars, you will pay $\left(\frac{0.0r}{12}\right)P$[1] dollars in interest on your first payment. Since you paid M dollars in total for the first payment, the remainder of the payment $(M-\left(\frac{r}{12}\right)P)$goes to pay down the principal. So, the principal remaining after the first payment (let's call it P_1) is the original principal minus what you paid on the principal, or

$$P_1 = P - \left(M - \left(\frac{r}{12}\right)P\right) = \left(1 + \frac{r}{12}\right)P - M$$

As long as P_1 is positive, you still have to keep making payments to pay off the loan.

a) Recall that the amount of interest you pay each time depends on the principal that remains. How much interest, in terms of P_1 and r, do you pay in the second installment?

(b) How much of your second monthly installment goes to pay off the principal? What is the principal P_2, or the balance of the loan, that you still have to pay off after making the second installment of the loan? Write your response in the form P_2= ( ) P_1 -( )M, where you fill in the parentheses.

c) Show that $P_2 = \left(1+\frac{r}{12}\right)^2 P - [1+\left(1+\frac{r}{12}\right)]M$.

d) Let P_3 be the amount of principal that remains after the third installment. Show that

[1] 0.0r changes the interest into decimal form.

$$P_2 = \left(1+\frac{r}{12}\right)^2 P - [1 + \left(1+\frac{r}{12}\right)]M$$

(e) If you continue in the manner described in the problems above, then the remaining principal of your loan after n installments is

$$P_n = \left(1+\frac{r}{12}\right)^n P - \left[\sum_{k=0}^{n-1}\left(1+\frac{r}{12}\right)^k\right] M$$

This is a rather complicated formula and one that is difficult to use. However, you can simplify the sum if you recognize part of it as a partial sum of a geometric series. Find a formula for the sum

$$\sum_{k=0}^{n-1}\left(1+\frac{r}{12}\right)^k$$

and then a general formula for P_n that does not involve a sum.

(f) It is usually more convenient to write your formula for P_n in terms of years rather than months. Show that P_t, the principal remaining after t years, can be written as

$$P(t) = \left(P - \frac{12M}{r}\right)\left(1+\frac{r}{12}\right)^{12t} + \frac{12M}{r}.$$

g) Now that you have analyzed the general loan situation, you apply formula in (f) to an actual loan. Suppose you charge $1,000 on a credit card for holiday expenses. If your credit card charges 20% interest and you pay only the minimum payment of $25 each month, how long will it take you to pay off the $1,000 charge? How much in total will you have paid on this $1,000 charge? How much total interest will you pay on this loan?

h) Now consider larger loans, e.g. automobile loans or mortgages, in which you borrow a specified amount of money over a specified period of time. In this situation, you need to determine the amount of the monthly payment you need to make to pay off the loan in the specified amount of time. In this situation, you need to find the monthly payment M that will take your outstanding principal to 0 in the specified amount of time. To do so, you want to know the value of M that makes $P(t) = 0$ in formula in (f). If you set $P(t) = 0$ and solve for M, it follows that

$$M = \frac{rP\left(1+\frac{r}{12}\right)^{12t}}{12\left(\left(1+\frac{r}{12}\right)^{12t}-1\right)}$$

.

i Suppose you want to borrow \$15,000 to buy a car. You take out a 5-year loan at 6.25%. What will your monthly payments be? How much in total will you have paid for this \$15,000 car? How much total interest will you pay on this loan?

ii Suppose you charge your books for winter semester on your credit card. The total charge comes to \$525. If your credit card has an interest rate of 18% and you pay \$20 per month on the card, how long will it take before you pay off this debt? How much total interest will you pay?

iii. Say you need to borrow \$100,000 to buy a house. You have several options on the loan:

– 30 years at 6.5%

– 25 years at 7.5%

– 15 years at 8.25%.

(a) What are the monthly payments for each loan?

(b) Which mortgage is ultimately the best deal (assuming you can afford the monthly payments)? In other words, for which loan do you pay the least amount of total interest?

VII. Formative Assessment – Khan Academy

Complete the following online practice exercises in the *Sequences and Series* unit of Khan Academy's AP Calculus BC course:

- https://www.khanacademy.org/math/ap-calculus-bc/series-bc/geo-series-bc/e/geometric-series--1

Optional:

- https://www.khanacademy.org/math/ap-calculus-bc/series-bc/partial-sums-bc/e/understanding-series

- https://www.khanacademy.org/math/ap-calculus-bc/series-bc/partial-sums-bc/e/convergence-and-divergence-of-series