4.2A Do Plants Know Math? (BC)

Arithmetic Series

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To everything there is a number. There is one you. Two eyes looking at this page. Three figures in the Christian trinity. Four legs on a chair. Five petals on the columbine flower. Six legs on insects. Seven is lucky. Eight counter-clockwise spirals of seeds on some pinecones. So many things to count. And from such counting, remarkable relationships and connections can emerge. Some are spurious, curiosities to the *numerologists* who use number mysticism as astrologers use the signs of the Zodiac. The *Pythagoreans*, the important sixth century B.C. sect of Greek mathematicians, and other important mathematicians have dabbled in numerology. Yet it is a subject short on substance, long on coincidence and happenstance.

Investigation 1: A Remarkable Sequence of Numbers

1a) How many rows of leaves are spiraling out from the picture of an aloe plant at the top of the page?

b) Using a marker, color one of the spiral arcs in the pinecone (pictured at left) that moves in a clockwise manner from the outer edge of the image to the center of the meristem. You will note that the spiral arc doesn't continue perfectly at the center of the meristem. Skip over the spiral that is adjacent to the one you just colored and color the next one that appears to have the same orientation after that. Continue this way around the pinecone until you have colored as many non-adjacent spiral arcs in the clockwise family as you can. How many clockwise spiral arcs are there?

In botany, a particularly compelling pattern of numbers emerges. When we count the number of petals on many different types of flowers, the number of spirals that appear on the surface textures of many fruits, and the arrangement of leaves on tree branches they usually do not find a random collection of numbers.

Rather, the numbers 5, 89, 13, 34, 8, 21, 55, 144… and 3 occur repeatedly and almost exclusively.

Arranged as they are above there might not seem to be anything striking about these numbers.

But, in numerical order the numbers form a clear pattern.

3 ,5, 8, 13, 21, 34, 55, 89, 144

c) Do you see a pattern? Describe the pattern in words.

d) What numbers would come next in the sequence?

e) What number(s) would come before the number "3"?

These numbers are called the **Fibonacci numbers**. Each number in the sequence is the sum of the two that come before it.

f) Write a recursive formula to find the nth term of the **Fibonacci** sequence.

g) What is the 73^{rd} Fibonacci number?

In the next Investigation, you will explore the genesis of the sequence: The sequence first appeared as a solution of a typically hokey word problem, one about rabbits, that appeared in an important mathematical text published in 1202 by a mathematician nicknamed Fibonacci.

Properly named **Leonardo of Pisa** (Italian mathematician; 1175 - 1250), this son of a well-known Italian merchant was better known as **Fibonacci** (a contraction of *filus Bonaccio*, "son of Bonaccio").

Fibonacci traveled widely as a student, learning methods of Arabic mathematics when studying in Northern Africa and learning the system of Hindu-Arabic numerals. Fibonacci assembled what he had learned into *Liber Abaci* (literally "book of the abacus", meaning book of arithmetic), the most comprehensive book of arithmetic of its time. It laid out the benefits of the Hindu-Arabic numeral system and is partially responsible for its wide acceptance subsequently. Fibonacci went on to publish several other books that focused mainly on arithmetic and algebra. These textbooks and his success in mathematical competitions in the court of Emperor Frederick II established him as the premier mathematician of the age.

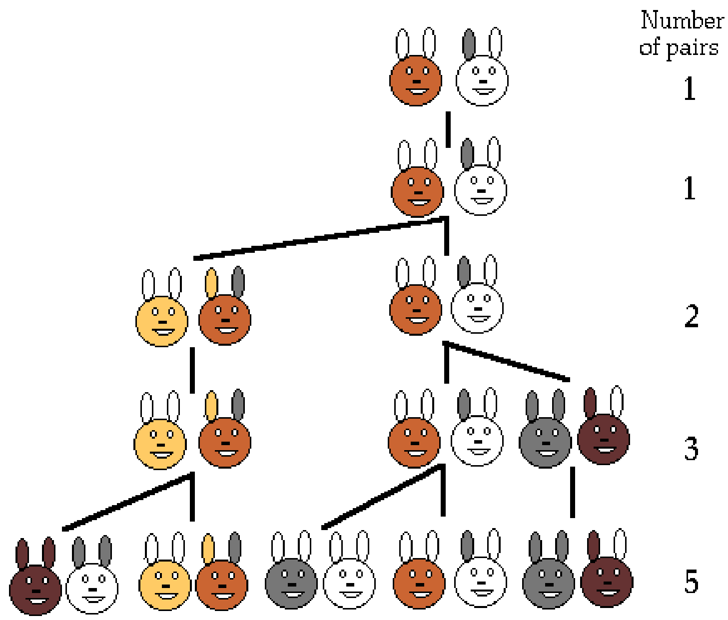
Investigation 2: Fibonacci's Problem

Despite his impact on the acceptance of the Hindu-Arabic numeral system, Fibonacci's most widespread notoriety comes from a single problem from among the hundreds that he used in Liber Abaci to illustrate the importance of the ideas laid out in this textbook. Fibonacci's famous problem was:

> How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on?

If you represent each pair of juvenile rabbits by *xy* and each pair of mature rabbits by *XY*, we can trace the number of rabbit pairs over the months as follows:

It was from this somewhat artificial word problem, not their appearance in nature, that the Fibonacci numbers were first discovered. Since their discovery they have, like breeding rabbits, flourished.



2a) Continue the breeding tree above for three more months, checking that it yields the next three Fibonacci numbers. (You might find it useful to use different colors rather than symbols to distinguish mature from juvenile rabbit pairs.)

b) Answer Fibonacci's question: how many pairs will be produced in a year?

II. An Introduction to Finite Series

While a sequence is simply a list of numbers, a ***series*** is created by adding terms in the sequence. There are two ways to indicate that you are adding terms in a sequence. One is by using summation notation and one is by using subscript notation, similar to what you've used write explicit forms of sequences.

In ***summation notation***, you are given an expression and told how many terms to add up. For example,

$$\sum_{n=1}^{3}(2n-1)$$

tells you to substitute the values of $n = 1$, $n = 2$, and $n = 3$ in the expression $2n$ - 1 and add them up.

$$\sum_{n=1}^{3}(2n-1) = (2\cdot 1-1)+(2\cdot 2-1)+(2\cdot 3-1)$$

$$= (1)+(3)+(5) = 9$$

The same series in subscript notation, would read $S_3 = 2n - 1$. Again, this simply means add all of the values of the expression when evaluated from $n = 1$ to $n = 3$.

Investigation 3: Sigma Notation

3a) Write the entire series that is defined by the following. You do not need to make the calculation!

i. $\sum_{n=1}^{5}(\frac{n}{2})$

ii. $\sum_{i=0}^{4}(n^2+3)$

iii. $\sum_{k=4}^{7}\sqrt{k+1}$

b) Write the following series using sigma notation. You do not need to make the calculation!

i. $2 + 3 + 4 + ... + 19 + 20$

ii. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{9}{10} + \frac{10}{11}$

iii. $e^3 + e^4 + e^5 + \cdots + e^{36} + e^{37}$

c) Calculate the series that is defined by the following

i. $\sum_{i=1}^{4}(i + 2)$

ii. $\sum_{n=2}^{5}(\frac{2n}{n-1})$

iii. $\sum_{i=0}^{4}(i^2\pi)$

IV. Formative Assessment – Khan Academy

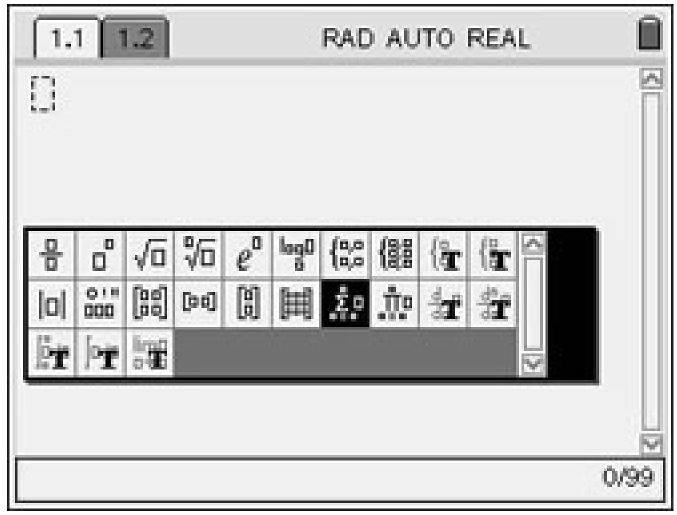
Complete the following online practice exercise in the *Sequences and Series* unit of Khan Academy's AP Calculus BC course:

- https://www.khanacademy.org/math/ap-calculus-bc/series-bc/series-tut-bc/e/evaluating-basic-sigma-notation

V. Calculating Finite Series

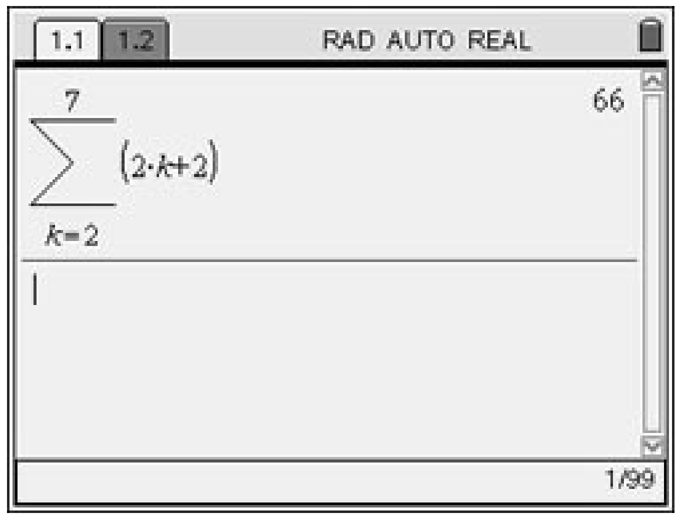
Most (all?) graphing calculators compute series. In the TI-Nspire, one way of quickly generating a blank sigma with empty boxes is the format button, as see below.

Complex series can be quickly calculated by entering the relevant values, as seen below.



Notice that any variable can be used in the argument below the sigma, but it must match the variable in the expression.

Investigation 4: Sigma Notation



4a) Calculate the defined series.

i. $\sum_{n=9}^{58}(\frac{n}{2})$

ii. $\sum_{i=0}^{214}(i^2 + 3)$

iii. $\sum_{k=4}^{41}\sqrt{k+1}$

b) Calculate the following series. Hint: before entering them into a calculator, you will need to determine an explicit formula that defines them.

i. $2 + 3 + 4 + \ldots + 19 + 20$

ii. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{9}{10} + \frac{10}{11}$

iii. $e^3 + e^4 + e^5 + \cdots + e^{36} + e^{37}$

iv. $1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55$

VI. Calculating Arithmetic Series… without a calculator!

Investigation 5:

5a) Watch the video (3:12) about Carl Gauss' method of calculating large sums. (Alternate site.)

b) Create a generalized formula for calculating the sum of an infinite series in terms of n.

c) Use your formula to calculate the defined series.

i. $\sum_{n=1}^{58}(n)$

ii. $\sum_{n=1}^{113}(2i+3)$

iii. $\sum_{k=1}^{23}(\frac{k}{2}+1)$

d) Consider the sequence of the first 20 even numbers,

2, 4, 6, 8, …,38, 40

Find the sum of the first 20 terms is denoted by S_{20}.

VI. Exercises

1. Consider the sequence of the first 40 odd numbers,

1, 3, 5, 7, …,37, 39

a) Find the sum of the first 30 terms is denoted by S_{30}.

2. Calculate the following series without a calculator.

-12 + (-8) + (-4) +…+ 52 + 56

3. find the sum of the first 125 terms of the following recursively defined sequence, without a calculator.

$a_i = a_{i-1} + 7, a_1 = 3$, for all $i \geq 2$

4) A student saves $3 dollars on August 1, $5 on August 2, $7 on August 3, and so on. How much will she save in August?

5) You take a job starting with an hourly rate of $16. You are given a raise of 25 cents per hour every 2 months for 5 years. What will your hourly wage be at the end of 5 years?

VII. Formative Assessment – Khan Academy

Complete the following online practice exercises in the *Sequences and Series* unit of Khan Academy's AP Calculus BC course:

- https://www.khanacademy.org/math/ap-calculus-bc/series-bc/series-tut-bc/e/arithmetic_series