# 4.1 All in a Row (BC)

Introduction to Series and Sequences

You encounter sequences every day. Your monthly rent payments, the annual interest you earn on investments, a list of your car's miles per gallon every time you fill up; all are examples of sequences. Other sequences with which you may be familiar include the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, \ldots$$

in which each entry is the sum of the two preceding entries and the triangular numbers
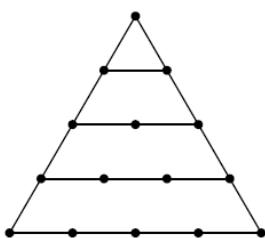
$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \ldots$$

which are numbers that correspond to the number of vertices seen in the triangles at left.

Sequences of integers are of such interest to mathematicians and others that they have a journal[1] devoted to them and an on-line encyclopedia[2] that catalogs a huge number of integer sequences and their connections. Sequences are also used in digital recordings and digital images.

To this point, most of your studies in calculus have dealt with continuous information (e.g., continuous functions). The major difference you will see now is that sequences model discrete instead of continuous information. You will study ways to represent and work with discrete information in this chapter as you investigate sequences and series, and ultimately see key connections between the discrete and continuous.

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[1] The Journal of Integer Sequences at http://www.cs.uwaterloo.ca/journals/JIS/

[2] The On-Line Encyclopedia of Integer Sequences at http://oeis.org/

Investigation 1: Suppose you receive $5000 through an inheritance. You decide to invest this money into a fund that pays 8% annually, compounded monthly. That means that each month your investment earns $\frac{0.08}{12} \cdot P$ additional dollars, where P is your principal balance at the start of the month. So, in the first month your investment earns

$$5000 \cdot \frac{0.08}{12}$$

or $33.33. If you reinvest this money, you will then have $5033.33 in your account at the end of the first month. From this point on, assume that you reinvest all of the interest you earn.

a) How much interest will you earn in the second month? How much money will you have in your account at the end of the second month?

b) Complete the table below to determine the interest earned and total amount of money in this investment each month for one year.

Month	Interest earned	Total amount of money in the account (EOM[3])
1	$ 0.00	$ 5000.00
2	$ 33.33	$ 5033.33
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		

c) As you will see later, the amount of money P_n in the account after month n is given by

$$P_n = 5000\left(1 + \frac{0.08}{12}\right)^n.$$

[3] EOM = end of month

d) Use this formula to check your calculations in (b). Then find the amount of money in the account after 5 years.

e) How many years will it be before the account has doubled in value to $10000?

II. Sequences

As the discussion in the introduction and Investigation 1 illustrate, many discrete phenomena can be represented as lists of numbers (like the amount of money in an account over a period of months). You call these any such list a sequence. In other words, a sequence is nothing more than list of terms in some order. To be able to refer to a sequence in a general sense, you often list the entries of the sequence with subscripts,

$$s_1, s_2, \ldots, s_n \ldots,$$

where the subscript denotes the position of the entry in the sequence. More formally,

> Definition 4.1. A sequence is a list of terms $s_1, s_2, s_3, \ldots$ in a specified order.

As an alternative to Definition 4.1, you can also consider a sequence to be a function f whose domain is the set of positive integers. In this context, the sequence $s_1, s_2, s_3, \ldots$ would correspond to the function f satisfying $f(n) = s_n$ for each positive integer n. This alternative view will be be useful in many situations.

You will often write the sequence

$$s_1, s_2, s_3, \ldots$$

using the shorthand notation s_n. The value s_n (alternatively $s(n)$) is called the nth term in the sequence. If the terms are all 0 after some fixed value of n, you say the sequence is finite. Otherwise the sequence is infinite. You will work with both finite and infinite sequences, but focus more on the infinite sequences. With infinite

sequences, you are often interested in their end behavior and the idea of *convergent* sequences.

Investigations 2-5: Exploring Sequences in the TI-Nspre

- Recursive calculations
- Explicit Formula
- Recurrence relations
- Plotting sequences

2. A linear sequence of numbers of numbers, such as 2, 5, 8, ... can be generated very easily in the Calculator App. The sequence 2, 5, 8, ... has an *initial term* 2. You then add 3 to get the next term.

a) Type 2 then press [Enter].

- Press [+] (this will paste an 'Ans') + 3.
- The question is now 'Ans + 3'. When you press [Enter], this is evaluated as '2 + 3', which returns 5.
- Pressing [Enter] again will re-evaluate the question (which is Ans + 3) as '5 + 3', giving an answer of 8.
- This can be continued as many times as needed, thus generating a linear sequence in a recursive fashion.

b) The sequence 4, 11, 32,... can be generated by starting with 4, then multiplying the previous term by 3 and subtracting 1. This is done on the calculator as shown:

- Starting number: Type 4, [Enter]
- To get next term: Press [×][4] '3' [-] '1'
- To get subsequent terms: Press [Enter] repeatedly

More complex sequences can be generated in a similar way, through the use of [Ctrl] + [Ans]. For example:

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[4] Multiplication sign

d) The sequence 5, 6, 13, 118, … can be generated by

- Starting number $(u_1) = 5$
- To get next term: Press $[x^2]$ [-] '4' $[\times]$ [Ctrl] [Ans]

e) What is the eighth term of the sequence generated below?

- Starting number (u_1): 7
- Next term: $[\div]$ 2 + 3 $[\times]$ [Ctrl] [Ans]

f) What is the 15th term (u_{15}) of the sequence generated below?

- Starting number (u_1): π
- Next term: $[\times]$ '3' $[\div]$ '2'

g) Write the *recursive* formula for the sequence 3, 5, 9, 17, 33, 65…

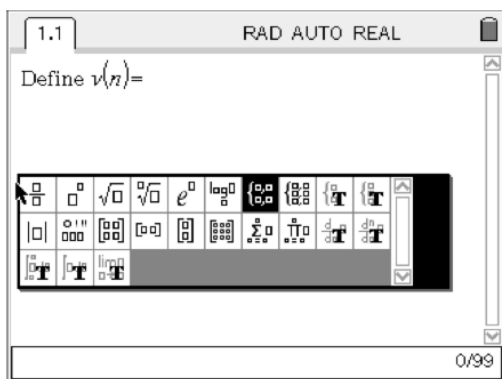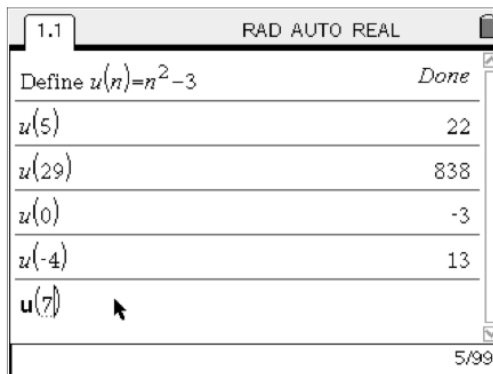h) Write the *explicit* formula for the sequence 3, 5, 9, 17, 33, 65…

2. With an explicit formula stored in the calculator, you can quickly evaluate the stored formula at various values of *n*.

- In Calculator mode, define your explicit formula:
    - Press: [Menu], [1. Actions], [1. Define]. and type $u(n) = n^2 - 3$

1.1   RAD AUTO REAL

Define $u(n)=n^2-3$ Done

1/99
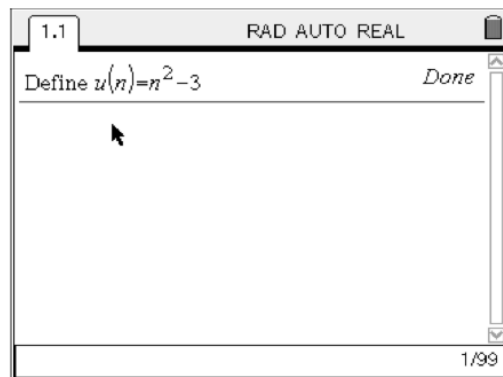
a) Evaluate the stored formula to find the value of the following terms:

> i.) $u(7)$
>
> ii.) $u(47)$
>
> iii.) $u(-3)$
>
> iv.) $u\left(\frac{2}{7}\right)$
>
> v.) $u(e)$

1.1	RAD AUTO REAL	
Define $u(n)=n^2-3$		Done
$u(5)$		22
$u(29)$		838
$u(0)$		-3
$u(-4)$		13
$u(7)$		
		5/99

| 1.1	RAD AUTO REAL	
Define $v(n)=$		

0/99

3. Defining a recursive formula in Calculator mode is slightly more complicated, as a piecewise function needs to be defined.

- Define the recurrence relation, v.
- Use the templates set up the piecewise function. The button is found next to the '9' key (pictured at right)

- In the formula entry bar, you now have space for the initial terms for a recursive formula, and the option to change values of n and the step size.

1.1	RAD AUTO REAL	
Define $v(n)=\begin{cases} 2, & n=1 \\ 3\cdot v(n-1), & n>1 \end{cases}$		Done
$v(1)$		2
$v(3)$		18
$v(10)$		39366
		4/99

- The initial condition must be in the first row of the piecewise function.
  - Define $v(n) =$
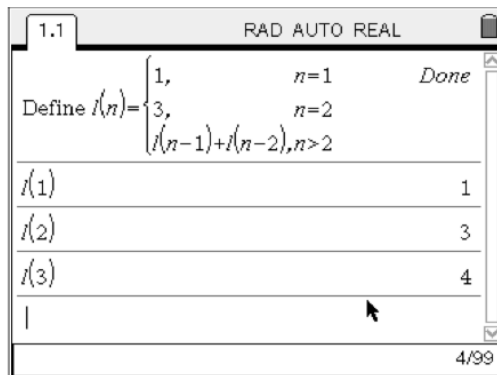$$\begin{cases} 2, & n=1 \\ 3 \cdot v(n-1) & n>1 \end{cases}$$
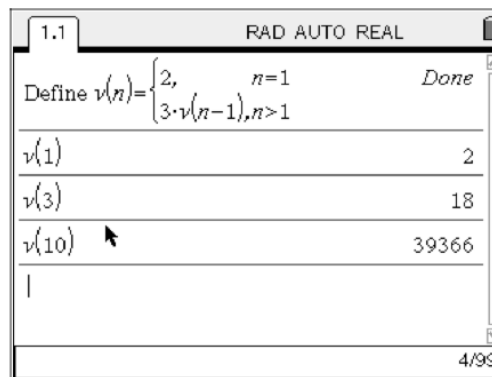
a) Evaluate the stored formula to find the value
of the following terms:

    i.) $v(6)$

    ii.) $v(13)$

    iii.) $v(21)$

b) fining a more complicated recurrence
relation, e.g. the Lucas sequence, can be
defined in the same way, with more rows in the piecewise definition. The initial
terms must be in ascending order.

c) Create your own sequence, define it below, and record the first 5 terms below

4. Plotting an explicit formula: In Graph mode :
(or in a Graph page) change the Graph Type to
Sequence:

- [Menu], [3. Graph Type], [5. Sequence], [1.
  Sequence]
- In the formula entry bar, enter
  $u1(n) = 0.2n + 3$.
- The initial term, $u1(1)$ should be left
  blank (you will need to delete the 0 that
  is there by default), unless you want the initial term to have a value different
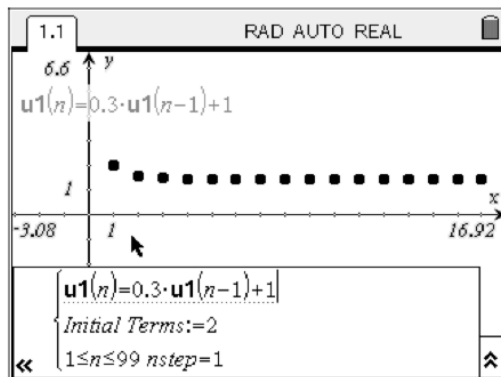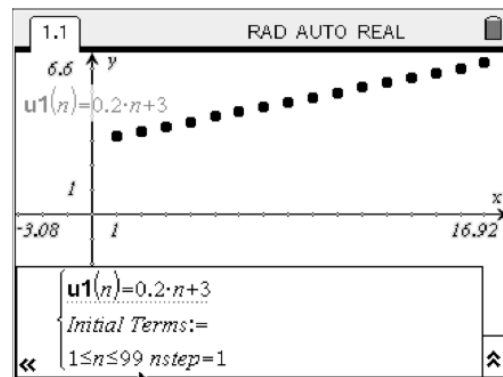 from what the formula would give.
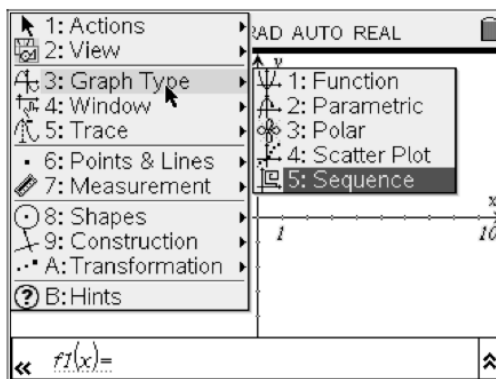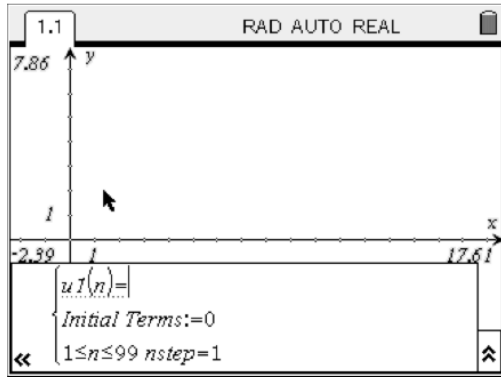
On pressing [ENTER], the formula entry bar will disappear. To bring it back, press [TAB], or [CTRL] + G.

To trace along the plot, choose Graph Trace: [MENU], [5:TRACE], [1:GRAPH TRACE].

Move left or right along the plot. Trace information is displayed in the bottom right corner as shown.

5. The plot of a recursive function is defined in a similar way as before. Be careful to set up the recursive part of the relation using the same notation as the left-hand side of the equation, i.e. use $u1(\ )$, u2( ), etc…

The values of n can also be changed from the default of $1 \leq n \leq 99$, by editing the 3rd row of the formula entry bar.
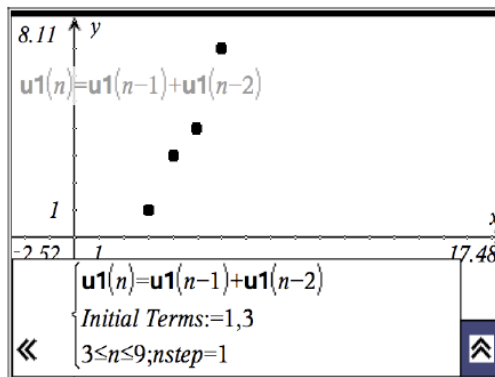
To avoid the calculator slowing down (spinning clock), consider reducing the size of the domain to, say, $1 \leq n \leq 10$.

The step size can also be changed to any value by editing the $n$step.

What is the effect of changing the domain of the function? – horizontal translation.

Follow the same steps to trace the plot, as for explicit formula.

III. Finding the nth Term of a Sequence

As seen in in Investigations 2-5, there are two ways to define a sequence – with a recursive formula or with an explicit formula. For each, as seen above, there are (at least) two ways of finding a specific term on a calculator:

- Start with the first term [Enter], enter the formula to get the next term, and keep hitting enter the required number of times, or
- Define a variable with the correct formula and enter the term number in $u(n)$ form.

Of course, you can use paper and pencil as well.

Investigations 6:

6a) Find the first five terms of the recursively defined sequence.

i. $a_1 = 4, \ a_n = a_{n-1} + 13,$ for all $n \geq 2$

ii. $b_1 = -3, \ b_n = b - 5,$ for all $n \geq 2$

iii. $c_1 = \frac{3}{7}, \ c_n = 2 \cdot c_{n-1} + \frac{1}{3},$ for all $n \geq 2$

iv. $d_1 = 11, \ d_n = d_{n-1} \div 1.1,$ for all $n \geq 2$

b) For each of the following sequences, identify the formula as recursive or explicit. Then use your handheld to find the specified term.

a. $a_n = n^2 - 3n$, find a_{37}

b. $b_1 = 2$, $b_n = 2 \cdot b_{n-1} - 3$, for all $n \geq 2$, find b_{21}

c. $c_n = \dfrac{n}{n+1}$, find c_{12}

d. $d_n = \left(1 + \dfrac{1}{n}\right)^n$, find d_{19}

e. $u_1 = 1, u_2 = 1$, $u_n = u_{n-1} + u_{n-2}$, for all $n \geq 2$, find u_{50}

f. $v_n = 3 - \dfrac{1}{n}$, find v_{42}

IV. Arithmetic and Geometric Sequences

There are many types of sequences, but two are dominant in mathematical applications: those in which pairs of successive terms all have a *common difference*, which are known as *arithmetic sequences*, and those in which pairs of successive terms all have a *common ratio, or geometric sequences*).

> Definition. A sequence $\{a_n\}$ is an **arithmetic sequence** if it can be written in the form
> $$\{a, a + d, \quad a + 2d, \dots, a + (n-1)d, \dots\}$$
> for some constant d. The number d is the **common difference**. Each term in an arithmetic sequence can be obtained recursively from its preceding term by adding d:
> $$a_n = a_{n-1} + d \text{ for all } n \geq 2.$$

Example 1.

For the following arithmetic sequences, find (a) the common difference, (b) a recursive rule for the nth term, and (c) an explicit rule for the nth term.

$$- 5, -2, 1, 4, 7, \dots$$

Solution.

(a) The difference between successive terms is 3.

(b) The sequence is defined recursively by $a_1 = -5, \ a_n = a_{n-1} + 3$, for all $n \geq 2$

(c) The sequence is defined explicitly by $a_n = -5 + (n-1)(3) = 3n - 8$.

Definition. A sequence {a$_n$} is a **geometric sequence** if it can be written in the form

$$\{a, ar, ar^2, \dots, ar^{n-1}, \dots\}$$

for some nonzero constant r. The number r is the **common ratio**.

Each term in a geometric sequence can be obtained recursively from its preceding term by multiplying by r:

$$a_n = a_{n-1} \cdot r \ \text{ for all } n \geq 2.$$

Example 2.

For the following geometric sequences, find (a) the common ratio, (b) a recursive rule for the nth term, and (c) an explicit rule for the nth term.

$$1, -2, 4, -8, 16, \dots$$

Solution.

(a) The ratio between successive terms is -2.

(b) The sequence is defined recursively by $a_1 = 1, \ a_n = (-2)a_{n-1}$, for all $n \geq 2$

(c) The sequence is defined explicitly by $a_n = (1)(-2)^{n-1} = (-2)^{n-1}$.

Investigations 7:

For the following sequences, (i.) identify the sequence as arithmetic or geometric, (ii.) state the common difference or ratio, (iii.) write a recursive rule for the nth term, and (iv.) write an explicit rule for the nth term.

a) 8, 4, 2, 1, …

b) -2, 1, 4, 7, …

c) 1, 1.5, 2.25, 3.375, …

d) $1, \frac{3}{2}, 2, \frac{5}{2}, …$

e) $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, 4\sqrt{2}, …$

f) -3, 9, -27, 81, …

V. Formative Assessment 4.1A

Complete the following online practice exercises in the *Sequences and Series Review* unit of Khan Academy's AP Calculus BC course:

1. https://www.khanacademy.org/math/ap-calculus-bc/series-bc/sequences-tut-bc/e/understanding-sequences

2. https://www.khanacademy.org/math/ap-calculus-bc/series-bc/sequences-tut-bc/e/geometric_sequences_1

3. https://www.khanacademy.org/math/ap-calculus-bc/series-bc/sequences-tut-bc/e/extend-geometric-sequences-negatives-fractions

4. https://www.khanacademy.org/math/ap-calculus-bc/series-bc/sequences-tut-bc/e/geometric_sequences_2

VI. Finding the Limit of a Sequence

In the following Investigation, you will examine which sequences have limits. As with functions, you can use a graphing calculator to suggest what a limiting value may be.

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Investigations 8:

8a) Let s_n be the nth term in the sequence 1, 2, 3, ...

Find a formula for s_n and use appropriate technological tools to draw a graph of entries in this sequence by plotting points of the form (n, s_n) for some values of n. Most graphing calculators can plot sequences;

Using your knowledge of limits of continuous functions as $x \to \infty$, decide if this sequence $\{s_n\}$ has a limit as $n \to \infty$. Explain your reasoning.

b) Let s_n be the nth term in the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ .... Find a formula for s_n. Draw a graph of some points in this sequence. Using your knowledge of limits of continuous functions as $x \to \infty$, decide if this sequence $\{s_n\}$ has a limit as $n \to \infty$. Explain your reasoning.

(c) Let s_n be the nth term in the sequence $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}$ .... Find a formula for s_n.

Using your knowledge of limits of continuous functions as $x \to \infty$, decide if this sequence $\{s_n\}$ has a limit as $n \to \infty$. Explain your reasoning.

Next you will formalize the ideas from Investigation 8.

Investigation 9:

a) Recall your earlier work with limits involving infinity. State clearly what it means for a continuous function f to have a limit L as $x \to \infty$.

b) Given that an infinite sequence of real numbers is a function from the integers to the real numbers, apply the idea from part (a) to explain what you think it means for a sequence $\{s_n\}$ to have a limit as $n \to \infty$.

c) Based on your response to (b), decide if the sequence $\left\{\frac{1+n}{2+n}\right\}$ a limit as $n \to \infty$. If so, what is the limit? If not, why not?

In Investigations 8 and 9 you investigated the notion of a sequence s_n having a limit as n goes to infinity. If a sequence s_n has a limit as n goes to infinity, you say that the sequence **converges** or is a **convergent sequence**. If the limit of a convergent sequence is the number L, you use the same notation as you did for continuous functions and write

$$\lim_{n \to \infty} s_n = L.$$

If a sequence s_n does not converge then you say that the sequence s_n **diverges**. Convergence of sequences is a major idea in this section and you describe it more formally as follows.

> A sequence $\{s_n\}$ of real numbers **converges** to a number L if you can make all values of s_k for $k \geq n$ as close to L as you want by choosing n to be sufficiently large.

Remember, the idea of sequence having a limit as $n \to \infty$ is the same as the idea of a continuous function having a limit as $x \to \infty$. The only new wrinkle here is that your sequences are discrete instead of continuous.

You conclude this lesson with a few more examples in the following activity.

Investigation 10: Use graphical and/or algebraic methods to determine whether each of the following sequences converges or diverges. If the sequence converges, find its limit.

a) $\left\{ \frac{1+2n}{3n-2} \right\}$

b) $a_n = \frac{5+3^n}{10+2^n}$

c) $a_n = n \sin\left(\frac{1}{n}\right)$

d) $\left\{ (-1)^n \left(\frac{n+7}{n^2+1}\right) \right\}$

e) $a_n = \frac{2n^5+3n^2-n+3}{23n^4-16n+1}$

f) $\left\{ \frac{10^n}{2+n!} \right\}$ (where ! is the **factorial** symbol and $n! = n(n-1)(n-2)\ldots(2)(1)$ for any positive integer n (as convention you define 0! to be 1)).

III. Exercises

1. Finding limits of convergent sequences can be a challenge. However, there is a useful tool you can adapt from your study of limits of continuous functions at infinity to use to find limits of sequences. You will explore with the sequence

$$\frac{\ln(n)}{n}$$

  a) Calculate the first 10 terms of this sequence. Based on these calculations, do you think the sequence converges or diverges? Why?

  b) For this sequence, there is a corresponding continuous function f defined by

$$f(x) = \frac{\ln(x)}{x}$$

  Draw the graph of $f(x)$ on the interval [0, 10] and then plot the entries of the sequence on the graph. What conclusion do you think you can draw about the

sequence $\left\{\frac{\ln(n)}{n}\right\}$ if $\lim\limits_{n\to\infty} f(x) = L$. Explain.

(c) Note that $f(x)$ has the indeterminate form $\frac{\infty}{\infty}$ as x goes to infinity. What idea from differential calculus can you use to calculate $\lim\limits_{n\to\infty} f(x) = L$.? Use this method to find $\lim\limits_{n\to\infty} f(x) = L$. What, then, is $\lim\limits_{n\to\infty} \frac{\ln(n)}{n}$.?

2. Return to the example begun in Investigation 1 to see how to derive the formula for the amount of money in an account at a given time. You do this in a general setting. Suppose you invest P dollars (called the *principal*) in an account paying $r\%$ interest compounded monthly. In the first month you will receive $\frac{r}{12}$ (here r is in decimal form; e.g., if you have 8% interest, you write $\frac{0.08}{12}$) of the principal P in interest, so you earn

$$P\left(\frac{r}{12}\right)$$

dollars in interest. Assume that you reinvest all interest. Then at the end of the first month your account will contain the original principal P plus the interest, or a total of

$$P_1 = P + P\left(\frac{r}{12}\right) = P\left(1 + \frac{r}{12}\right)$$

dollars.

a) Given that your principal is now P_1 dollars, how much interest will you earn in the second month? If P_2 is the total amount of money in your account at the end of the second month, explain why

$$P_2 = P_1\left(1 + \frac{r}{12}\right) = P\left(1 + \frac{r}{12}\right)^2$$

b) Find a formula for P_3, the total amount of money in the account at the end of the third month in terms of the original investment P.

c) There is a pattern to these calculations. Let P_n the total amount of money in

the account at the end of the third month in terms of the original investment P. Find a formula for P_n.

3. Sequences have many applications in mathematics and the sciences. In a recent paper[5] the authors write

> *The incretin hormone glucagon-like peptide-1 (GLP-1) is capable of ameliorating glucose-dependent insulin secretion in subjects with diabetes. However, its very short half-life (1.5-5 min) in plasma represents a major limitation for its use in the clinical setting.*

The half-life of GLP-1 is the time it takes for half of the hormone to decay in its medium. For this exercise, assume the half-life of GLP-1 is 5 minutes. So if A is the amount of GLP-1 in plasma at some time t, then only $\frac{A}{2}$ of the hormone will be present after $t + 5$ minutes. Suppose $A_0 = 100$ grams of the hormone are initially present in plasma.

a) Let A_1 be the amount of GLP-1 present after 5 minutes. Find the value of A_1.

b) Let A_2 be the amount of GLP-1 present after 10 minutes. Find A_2.

c) Let A_3 be the amount of GLP-1 present after 15 minutes. Find A_3.

d) Let A_4 be the amount of GLP-1 present after 20 minutes. Find A_4.

e) Let A_n be the amount of GLP-1 present after $5n$ minutes. Find a formula for A_n.

f) Does the sequence A_n converge or diverge? If the sequence converges, find its limit and explain why this value makes sense in the context of this problem.

g) Determine the number of minutes it takes until the amount of GLP-1 in plasma is 1 gram.

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[5] Hui H, Farilla L, Merkel P, Perfetti R. The short half-life of glucagon-like peptide-1 in plasma does not reflect its long-lasting beneficial effects, Eur J Endocrinol 2002 Jun;146(6):863-9.

4. Continuous data is the basis for analog information, like music stored on old cassette tapes or vinyl records. A digital signal like on a CD or MP3 file is obtained by sampling an analog signal at some regular time interval and storing that information. For example, the sampling rate of a compact disk is 44,100 samples per second. So a digital recording is only an approximation of the actual analog information. Digital information can be manipulated in many useful ways that allow for, among other things, noisy signals to be cleaned up and large collections of information to be compressed and stored in much smaller space. While you won't investigate these techniques in this chapter, this exercise is intended to give an idea of the importance of discrete (digital) techniques.

Let f be the continuous function defined by $f(x) = \sin 4x$ on the interval $[0, 10]$. A graph of f is shown in the graph below. You approximate f by sampling, that is by partitioning the interval $[0, 10]$ into uniform subintervals and recording the values of f at the endpoints.
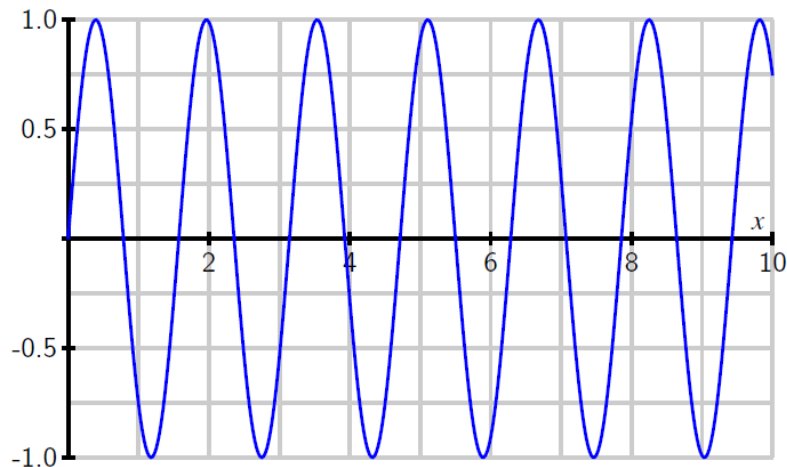
a) Ineffective sampling can lead to several problems in reproducing the original signal. As an example, partition the interval $[0, 10]$ into 8 equal length subintervals and create a list of points (the **sample**) using the endpoints of each subinterval. Plot your sample on graph of f above. What can you say about the period of your sample as compared to the period of the original function?

(b) The sampling rate is the number of samples of a signal taken per second. As part (a) illustrates, sampling at too small a rate can cause serious problems with reproducing the original signal (this problem of inefficient sampling leading to an inaccurate approximation is called **aliasing**). There is an elegant theorem

called the Nyquist-Shannon Sampling Theorem that says that human perception is limited, which allows that replacement of a continuous signal with a digital one without any perceived loss of information. This theorem also provides the lowest rate at which a signal can be sampled (called the Nyquist rate) without such a loss of information. The theorem states that you should sample at double the maximum desired frequency so that every cycle of the original signal will be sampled at least two points.

Recall that the frequency of a sinusoidal function is the reciprocal of the period. Identify the frequency of the function f and determine the number of partitions of the interval [0, 10] that give you the Nyquist rate.

c) Humans cannot typically pick up signals above 20 kHz. Explain why, then, that information on a compact disk is sampled at 44,100 Hz.

V. Formative Assessment 4.1B

Complete the following online practice exercises in the *Sequences and Series* unit of Khan Academy's AP Calculus BC course:

- https://www.khanacademy.org/math/ap-calculus-bc/series-bc/seq-conv-diverg-bc/e/convergence-and-divergence-of-sequences