2.36 Related Rates

Related Rates

In most of your applications of the derivative so far, you have worked in settings where one quantity (often called y) depends explicitly on another (say x), and in some way you have been interested in the instantaneous rate at which y changes with respect to x, leading you to compute $\frac{dy}{dx}$. These settings emphasize how the derivative enables you to quantify how the quantity y is changing as x changes at a given x-value.

You are next going to consider situations where multiple quantities are related to one another and changing, but where each quantity can be considered an implicit function of the variable t, which represents time. Through knowing how the quantities are related, you will be interested in determining how their respective rates of change with respect to time are related.

Investigation 1: A spherical balloon is being inflated at a constant rate of 20 cubic inches per second. How fast is the radius of the balloon changing at the instant the balloon's diameter is 12 inches? Is the radius changing more rapidly when $d = 12$ or when $d = 16$? Why?

a) Draw several spheres with different radii, and observe that as volume changes, the radius, diameter, and surface area of the balloon also change.

b) Recall that the volume of a sphere of radius r is $V = \pi r^3$. Note well that in the setting of this problem, both V and r are changing as time t changes, and thus both V and r may be viewed as implicit functions of t, with respective derivatives $\frac{dV}{dt}$ and $\frac{dr}{dt}$.

Differentiate both sides of the equation $V = \pi r^3$ with respect to t (using the chain rule on the right) to find a formula for $\frac{dV}{dt}$ that depends on both r and $\frac{dr}{dt}$.

c) At this point in the problem, by differentiating you have "related the rates" of change of V and r. Recall that you are given in the problem that the balloon is being inflated at a constant rate of 20 cubic inches per second. Is this rate the value of $\frac{dV}{dt}$ or $\frac{dr}{dt}$? Why?

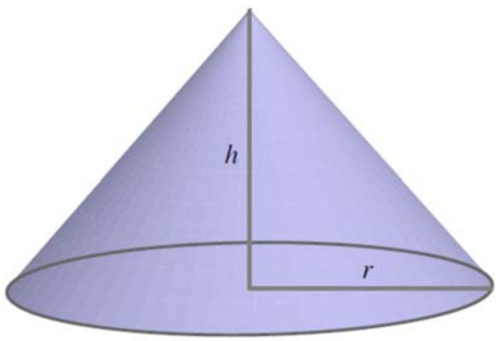
d) From part (c), you know the value of $\frac{dr}{dt}$ at every value of t. Next, observe that when the diameter of the balloon is 12, you know the value of the radius. In the equation $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$, substitute these values for the relevant quantities and solve for the remaining unknown quantity, which is $\frac{dr}{dt}$. How fast is the radius changing at the instant $d = 12$?

e) How is the situation different when $d = 16$? When is the radius changing more rapidly, when $d = 12$ or when $d = 16$?

II. Related Rates Problems

In problems where two or more quantities can be related to one another, and all of the variables involved can be viewed as implicit functions of time, t, you are often interested in how the rates of change of the individual quantities with respect to time are themselves related; you call these related rates problems. Often these problems involve identifying one or more key underlying geometric relationships to relate the variables involved. Once you have an equation establishing the fundamental relationship among variables, you differentiate implicitly with respect to time to find connections among the rates of change.

Example 1. Consider the situation where sand is being dumped by a conveyor belt on a pile so that the sand forms a right circular cone, as pictured below As sand falls from the conveyor belt onto the top of the pile, obviously several features of the sand pile will change: the volume of the pile will grow, the height will increase, and the radius will get bigger, too. All of these quantities are related to one another, and the rate at which each is changing is related to the rate at which sand falls from the conveyor.



The first key steps in any related rates problem involve identifying which variables are changing and how they are related. In the current problem involving a conical pile of sand, you observe that the radius and height of the pile are related to the volume of the pile by the standard equation for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$

Viewing each of V, r, and h as functions of t, you can differentiate implicitly to determine an equation that relates their respective rates of change. Taking the derivative of each side of the equation with respect to t,

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{1}{3}\pi r^2 h\right]$$

On the left, $\frac{d}{dt}[V]$ is simply $\frac{dV}{dt}$. On the right, the situation is more complicated, as both r and h are implicit functions of t, hence you have to use the product and chain rules. Doing so, you find that

$$\frac{dV}{dt} = \frac{d}{dt}\left[\frac{1}{3}\pi r^2 h\right]$$

$$= \frac{1}{3}\pi r^2 \frac{d}{dt}[h] + \frac{1}{3}\pi h \frac{d}{dt}[r^2]$$

$$= \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{1}{3}\pi h 2r \frac{dr}{dt}$$

You have now related the rates of change of V, h, and r. If you are given sufficient information, you may then find the value of one or more of these rates of change at one or more points in time. Say, for instance, that you know the following: (a) sand falls from the conveyor in such a way that the height of the pile is always half the radius, and (b) sand falls from the conveyor belt at a constant rate of 10 cubic feet per minute. With this information given, you can answer questions such as: how fast is the height of the sand pile changing at the moment the radius is 4 feet?

The information that the height is always half the radius tells us that for all values of t, $h = \frac{1}{2}r$. Differentiating with respect to t, it follows that $\frac{dh}{dt} = \frac{1}{2}\frac{dr}{dt}$. These relationships enable you to relate $\frac{dV}{dt}$ exclusively to just one of r or h. Substituting the expressions involving r and $\frac{dr}{dt}$ for h and $\frac{dh}{dt}$, you now have that

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \cdot \frac{1}{2}\frac{dr}{dt} + \frac{2}{3}\pi r \cdot \frac{1}{2}r \cdot \frac{dr}{dt}$$

Since sand falls from the conveyor at the constant rate of 10 cubic feet per minute, this tells you the value of $\frac{dV}{dt}$, the rate at which the volume of the sand pile changes. In particular, $\frac{dV}{dt} = 10$ ft^3/min. Furthermore, since you are interested in how fast the height of the pile is changing at the instant $r = 4$, you use the value $r = 4$ along with $\frac{dV}{dt} = 10$, and hence find that

$$10 = \frac{1}{3}\pi r^2 \cdot \frac{1}{2}\frac{dr}{dt}\bigg|_{r=4} + \frac{2}{3}\pi r \cdot \frac{1}{2}r \cdot \frac{dr}{dt}\bigg|_{r=4}$$

$$10 = \frac{8}{3}\pi \frac{dr}{dt}\bigg|_{r=4} + \frac{16}{3}\pi \frac{dr}{dt}\bigg|_{r=4}$$

With only the value of $\frac{dr}{dt}\Big|_{r=4}$ remaining unknown, you solve for $\frac{dr}{dt}\Big|_{r=4}$

and find that

$$10 = 8\pi \frac{dr}{dt}\bigg|_{r=4}$$

, so that

$$\frac{dr}{dt}\bigg|_{r=4} = \frac{10}{8\pi} \approx 0.398\ {}^{\text{ft}}/_{\text{sec}}$$

Because you were interested in how fast the height of the pile was changing at this instant, you want to know $\frac{dh}{dt}$ when r = 4. Since $\frac{dh}{dt} = \frac{1}{2}\frac{dr}{dt}$ for all values of t, it follows

$$\left.\frac{dr}{dt}\right|_{r\,=\,4} = \frac{5}{8\pi} \approx 0.199\ \text{ft}/\text{sec}$$

Note particularly how you distinguish between the notations $\frac{dh}{dt}$ and $\left.\frac{dr}{dt}\right|_{r\,=\,4}$.

The former is the rate of change of *r* with respect to *t* at an arbitrary value of *t*, while the latter is the rate of change of *r* with respect to *t* at a particular moment, in fact the moment $r = 4$. While we don't know the exact value of *t*, because information is provided about the value of *r*, it is important to distinguish that you are using this more specific data.

The relationship between *h* and *r*, with $h = \frac{1}{2}r$ for all values of *t*, enables you to transition easily between questions involving *r* and *h*. Indeed, had you known this information at the problem's outset, you could have immediately simplified your work. Using $h = \frac{1}{2}r$, it follows that since $V = \frac{1}{3}\pi r^2 h$, you can write *V* solely in terms of *r* to have

$$V = \frac{1}{3}\pi r^2 \cdot \frac{1}{2}r = \frac{1}{6}\pi r^3$$

From this last equation, differentiating with respect to *t* implies

$$\frac{dV}{dt} = \frac{1}{2}\pi r^2 \frac{dr}{dt}$$

from which the same conclusions made earlier about $\frac{dr}{dt}$ and $\frac{dh}{dt}$ can be made.

Here is summary of the main approach for subsequent problems.

- Identify the quantities in the problem that are changing and choose clearly defined variable names for them. Draw one or more figures that clearly represent the situation.
- Determine all rates of change that are known or given and identify the rate(s) of change to be found.
- Find an equation that relates the variables whose rates of change are known to those variables whose rates of change are to be found.
- Differentiate implicitly with respect to t to relate the rates of change of the involved quantities.
- Evaluate the derivatives and variables at the information relevant to the instant at which a certain rate of change is sought. Use proper notation to

identify when a derivative is being evaluated at a particular instant, such as $\left.\frac{dr}{dt}\right|_{r=4}$.

Sometimes a sequence of pictures can be helpful; for some already-drawn pictures that can be easily modified as applets built in Geogebra, see the following links which represent

- how a circular oil slick's area grows as its radius increases http://gvsu.edu/s/9n
- how the location of the base of a ladder and its height along a wall change as the ladder slides http://gvsu.edu/s/9o

Drawing well-labeled diagrams and envisioning how different parts of the figure change is a key part of understanding related rates problems and being successful at solving them.

Investigation 2: A water tank has the shape of an inverted circular cone (point down) with a base of radius 6 feet and a depth of 8 feet. Suppose that water is being pumped into the tank at a constant instantaneous rate of 4 cubic feet per minute.

(A similar problem is discussed and pictured dynamically at http://gvsu.edu/s/9p Exploring the applet at the link will be helpful to you in answering the questions that follow.)

a) Draw a picture of the conical tank, including a sketch of the water level at a point in time when the tank is not yet full. Introduce variables that measure the radius of the water's surface and the water's depth in the tank, and label them on your figure.

b) Say that *r* is the radius and *h* the depth of the water at a given time, *t*. What equation relates the radius and height of the water, and why?

c) Determine an equation that relates the volume of water in the tank at time *t* to the depth *h* of the water at that time.

d) Through differentiation, find an equation that relates the instantaneous rate of change of water volume with respect to time to the instantaneous rate of change of water depth at time *t*.

e) Find the instantaneous rate at which the water level is rising when the water in the tank is 3 feet deep.

f) When is the water rising most rapidly: at $h = 3$, $h = 4$, or $h = 5$?

Recognizing familiar geometric configurations is one way that we relate the changing quantities in a given problem. For instance, while the problem in Investigation 2 is centered on a conical tank, one of the most important observations is that there are two key right triangles present. In another setting, a right triangle might be indicative of an opportunity to take advantage of the Pythagorean Theorem to relate the legs of the triangle. But in the conical tank, the fact that the water at any time fills a portion of the tank in such a way that the ratio of radius to depth is constant turns out to be the most important relationship with which to work. That enables you to write r in terms of h and reduce the overall problem to one that involves only one variable, where the volume of water depends simply on h, and hence to subsequently relate $\frac{dV}{dt}$ and $\frac{dh}{dt}$. In other situations, where a changing angle is involved, a right triangle may offer the opportunity to find relationships among various parts of the triangle using trigonometric functions.

Investigation 3: A television camera is positioned 4000 feet from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. In addition, the auto-focus of the camera has to take into account the increasing distance between the camera and the rocket. You assume that the rocket rises vertically. (A similar problem is discussed and pictured dynamically at http://gvsu.edu/s/9t Exploring the applet at the link will be helpful to you in answering the questions that follow.)

a) Draw a figure that summarizes the given situation. What parts of the picture are changing? What parts are constant? Introduce appropriate variables to represent the quantities that are changing.

b) Find an equation that relates the camera's angle of elevation to the height of the rocket, and then find an equation that relates the instantaneous rate of change of the camera's elevation angle to the instantaneous rate of change of the rocket's height (where all rates of change are with respect to time).

c) Find an equation that relates the distance from the camera to the rocket to the rocket's height, as well as an equation that relates the instantaneous rate of change of distance from the camera to the rocket to the instantaneous rate of change of the rocket's height (where all rates of change are with respect to time).

d) Suppose that the rocket's speed is 600 ft/sec at the instant it has risen 3000 feet. How fast is the distance from the television camera to the rocket changing at that moment? If the camera is following the rocket, how fast is the camera's angle of elevation changing at that same moment?

e) If from an elevation of 3000 feet onward the rocket continues to rise at 600 feet/sec, will the rate of change of distance with respect to time be greater when the elevation is 4000 feet than it was at 3000 feet, or less? Why?

Investigation 4: As pictured in the applet at http://gvsu.edu/s/9q a skateboarder who is 6 feet tall rides under a 15 foot tall lamppost at a constant rate of 3 feet per second. You are interested in understanding how fast his shadow is changing at various points in time.

a) Draw an appropriate right triangle that represents a snapshot in time of the skateboarder, lamppost, and his shadow. Let x denote the horizontal distance from the base of the lamppost to the skateboarder and s represent the length of his shadow. Label these quantities, as well as the skateboarder's height and the lamppost's height on the diagram.

b) Observe that the skateboarder and the lamppost represent parallel line segments in the diagram, and thus similar triangles are present. Use similar triangles to establish an equation that relates x and s.

c) Use your work in (b) to find an equation that relates $\frac{dx}{dt}$ and $\frac{ds}{dt}$.

d) At what rate is the length of the skateboarder's shadow increasing at the instant the skateboarder is 8 feet from the lamppost?

e) As the skateboarder's distance from the lamppost increases, is his shadow's length increasing at an increasing rate, increasing at a decreasing rate, or increasing at a constant rate?

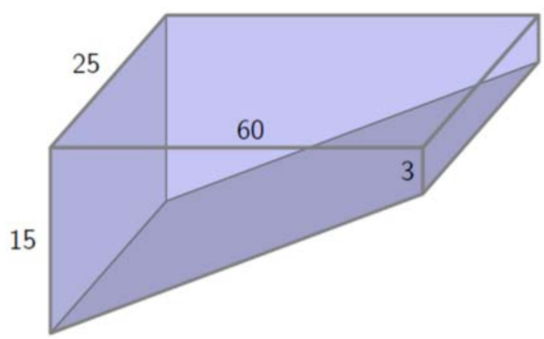
f) Which is moving more rapidly: the skateboarder or the tip of his shadow?

Explain, and justify your answer.

III. Exercises

1. A sailboat is sitting at rest near its dock. A rope attached to the bow of the boat is drawn in over a pulley that stands on a post on the end of the dock that is 5 feet higher than the bow. If the rope is being pulled in at a rate of 2 feet per second, how fast is the boat approaching the dock when the length of rope from bow to pulley is 13 feet?

2. A swimming pool is 60 feet long and 25 feet wide. Its depth varies uniformly from 3 feet at the shallow end to 15 feet at the deep end, as shown below.



Suppose the pool has been emptied and is now being filled with water at a rate of 800 cubic feet per minute.

At what rate is the depth of water (measured at the deepest point of the pool) increasing when it is 5 feet deep at that end?

Over time, describe how the depth of the water will increase: at an increasing rate, at a decreasing rate, or at a constant rate. Explain.

3. Sand is being dumped off a conveyor belt onto a pile in such a way that the pile forms in the shape of a cone whose radius is always equal to its height. Assuming that the sand is being dumped at a rate of 10 cubic feet per minute, how fast is the height of the pile changing when there are 1000 cubic feet on the pile?

4. A baseball diamond is 90, square. A batter hits a ball along the third base line and runs to first base. At what rate is the distance between the ball and first base changing when the ball is halfway to third base, if at that instant the ball is traveling 100 feet/sec? At what rate is the distance between the ball and the runner changing at the same instant, if at the same instant the runner is 1/8 of the way to first base

running at 30 feet/sec?

IV. Assessments – Khan Academy

1. Complete the first ten online practice exercises in the Analyzing Functions unit (Differentiating Common Functions) of Khan Academy's AP Calculus AB course:
   a. https://www.khanacademy.org/math/ap-calculus-ab/derivative-applications-ab/related-rates-ab/e/related-rates-multiple-rates
   b. https://www.khanacademy.org/math/ap-calculus-ab/derivative-applications-ab/related-rates-ab/e/related-rates-pythagorean-theorem
   c. https://www.khanacademy.org/math/ap-calculus-ab/derivative-applications-ab/related-rates-ab/e/related-rates-advanced
2. Optional: none