### 2.35 Get Real

Applied Optimization Problems

You are now ready to apply the ideas of calculus to determine and justify the absolute minimum or maximum. Thus, what is primarily different about
 problems of this type is that the problem-solver must do considerable work to introduce variables and develop the correct function and domain to represent the described situation.

Investigation 1: According to U.S. postal regulations, the girth plus the length of a parcel sent by mail may not exceed 108 inches, where by "girth" you mean the perimeter of the smallest end. What is the largest possible volume of a rectangular parcel with a square end that can be sent by mail? What are the dimensions of the package of largest volume?

a) Let $x$ represent the length of one side of the square end and $y$ the length of the longer side. Label these quantities appropriately on the image shown at left.
b) What is the quantity to be optimized in this problem? Find a formula for this quantity in terms of $x$ and $y$.
c) The problem statement tells you that the parcel's girth plus length may not exceed 108 inches. In order to maximize volume, you assume that you will actually need the girth plus length to equal 108 inches. What equation does this produce involving $x$ and $y$ ?
d) Solve the equation you found in (c) for one of $x$ or $y$ (whichever is easier).
e) Now use your work in (b) and (d) to determine a formula for the volume of the parcel so that this formula is a function of a single variable.
f) Over what domain should you consider this function? Note that both $x$ and $y$ must be positive; how does the constraint that girth plus length is 108 inches produce intervals of possible values for $x$ and $y$ ?
g) Find the absolute maximum of the volume of the parcel on the domain you established in (f) and hence also determine the dimensions of the box of greatest volume. Justify that you've found the maximum using calculus.

## II. More applied optimization problems

Many of the steps in Investigation 1 are ones that you will execute in any applied optimization problem. Those steps are summarized :

- Draw a picture and introduce variables. It is essential to first understand what quantities vary in the problem and then to represent those values with variables. Constructing a figure with the variables labeled is almost always an essential first step. Sometimes drawing several diagrams can be especially helpful.
- Identify the quantity to be optimized as well as any key relationships among the variable quantities. Essentially this step involves writing equations that involve the variables that have been introduced: one to represent the quantity
whose minimum or maximum is sought, and possibly others that show how multiple variables in the problem may be interrelated.
- Determine a function of a single variable that models the quantity to be optimized; this may involve using other relationships among variables to eliminate one or more variables in the function formula.
- Decide the domain on which to consider the function being optimized. Often the physical constraints of the problem will limit the possible values that the independent variable can take on. Thinking back to the diagram describing the overall situation and any relationships among variables in the problem often helps identify the smallest and largest values of the input variable.
- Use calculus to identify the absolute maximum and/or minimum of the quantity being optimized. This always involves finding the critical numbers of the function first. Then, depending on the domain, you either construct a first derivative sign chart (for an open or unbounded interval) or evaluate the function at the endpoints and critical numbers (for a closed, bounded interval).
- Finally, make certain you have answered the question and include appropriate units: does the question seek the absolute maximum of a quantity, or the values of the variables that produce the maximum? That is, finding the absolute maximum volume of a parcel is different from finding the dimensions of the parcel that produce the maximum.

Investigation 2: A soup can in the shape of a right circular cylinder is to be made from two materials. The material for the side of the can costs $\$ 0.015$ per square inch and the material for the lids costs $\$ 0.027$ per square inch. Suppose that you desire to construct a can that has a volume of 16 cubic inches. What dimensions minimize the cost of the can?
a) Draw a picture of the can and label its dimensions with appropriate variables.
b) Use your variables to determine expressions for the volume, surface area, and cost of the can.
c) Determine the total cost function as a function of a single variable. What is the domain on which you should consider this function?
(d) Find the absolute minimum cost and the dimensions that produce this value.

Familiarity with common geometric formulas is particularly helpful in problems like the previous one. Sometimes those involve perimeter, area, volume, or surface area. At other times, the constraints of a problem introduce right triangles (where the Pythagorean Theorem applies) or other functions whose formulas provide relationships among variables present.

Investigation 3: A hiker starting at a point $P$ on a straight road walks east towards point $Q$, which is on the road and 3 kilometers from point $P$. Two kilometers due north of point $Q$ is a cabin. The hiker will walk down the road for a while, at a pace of 8 kilometers per hour. At some point $Z$ between $P$ and $Q$, the hiker leaves the road and makes a straight line towards the cabin through the woods, hiking at a pace of 3 kph , as pictured at the right. To minimize the time to go from $P$ to $Z$ to the cabin, where should the hiker turn into the forest?


Investigation 4: Consider the region in the $x-y$ plane that is bounded by the $x$-axis and the function $f(x)=25-x^{2}$. Construct a rectangle whose base lies on the $x$-axis and is centered at the origin, and whose sides extend vertically until they intersect the curve $f(x)=25-x^{2}$. Which such rectangle has the maximum possible area? Which such rectangle has the greatest perimeter? Which has the greatest combined perimeter and area?

Investigation 4: A trough is being constructed by bending a $4 \times 24$ (measured in feet) rectangular piece of sheet metal. Two symmetric folds 2 feet apart will be made parallel to the longest side of the rectangle so that the trough has cross-sections in the shape of a trapezoid, as pictured below. At what angle should the folds be made to produce the trough of maximum volume?


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## III. Exercises

1. A rectangular box with a square bottom and closed top is to be made from two materials. The material for the side costs $\$ 1.50$ per square foot and the material for the bottom costs $\$ 3.00$ per square foot. If you are willing to spend $\$ 15$ on the box, what is the largest volume it can contain? Justify your answer completely using calculus.
2. A farmer wants to start raising cows, horses, goats, and sheep, and desires to have a rectangular pasture for the animals to graze in. However, no two different kinds of animals can graze together. To minimize the amount of fencing she will need, she has decided to enclose a large rectangular area and then divide it into four equally sized pens by adding three segments of fence inside the large rectangle that are parallel to two existing sides. She has decided to purchase 7500 ft of fencing. What is the maximum possible area that each of the four pens will enclose?
3. Two vertical poles of heights 60 ft and 80 ft stand on level ground, with their bases 100 ft apart. A cable that is stretched from the top of one pole to some point on the ground between the poles, and then to the top of the other pole. What is the minimum possible length of cable required? Justify your answer completely using calculus.
4. A company is designing propane tanks that are cylindrical with hemispherical ends. Assume that the company wants tanks that will hold 1000 cubic feet of gas, and that the ends are more expensive to make, costing $\$ 5$ per square foot, while the cylindrical barrel between the ends costs $\$ 2$ per square foot. Use calculus to determine the minimum cost to construct such a tank.
