### 2.33 All in the Family (BC)

Using Parameters to Describe Families of Functions

Mathematicians are often interested in making general observations, say
 by describing patterns that hold in a large number of cases. For example, think about the Pythagorean Theorem: it doesn't tell you something about a single right triangle, but rather a fact about every right triangle, thus providing key information about every member of the right triangle family. In this lesson, you would like to use calculus to help you make general observations about families of functions that depend on one or more parameters. People who use applied mathematics, such as engineers and economists, often encounter the same types of functions in various settings where only small changes to certain constants occur. These constants are called parameters.

You are already familiar with certain families of functions. For example, $f(t)=a \sin (b(t-c)+d$ is a stretched and shifted version of the sine function with amplitude a, period $\frac{2 \pi}{b}$, phase shift $c$, and vertical shift $d$. You understand from experience with trigonometric functions that a affects the size of the oscillation, $b$ the rapidity of oscillation, and c where the oscillation starts, as shown in the graph at
 left, while $d$ affects the vertical positioning of the graph.

In addition, there are several basic situations that you already understand completely. For instance, every function of the form $y=m x+b$ is a line with slope $m$ and $y$-intercept $(0, b)$
. Note that the form $y=m x+b$ allows you to consider every possible line
by using two parameters (except for vertical lines which are of the form $x=a$ ). Further, you understand that the value of $m$ affects the line's steepness and whether the line rises or falls from left to right, while the value of $b$ situates the line vertically on the coordinate axes.

For other less familiar families of functions, you would like to use calculus to understand and classify where key behavior occurs: where members of the family are increasing or decreasing, concave up or concave down, where relative extremes occur, and more, all in terms of the parameters involved.

Investigation 1: Let $a, h$, and $k$ be arbitrary real numbers with $a \neq 0$, and let $f$ be the function given by the rule $f(x)=a(x-h)^{2}+k$.
a) What familiar type of function is $f$ ? What information do you know about $f$ just by looking at its form? (Think about the roles of $a, h$, and $k$.)
b) Next you use some calculus to develop familiar ideas from a different perspective. To start, treat $a, h$, and $k$ as constants and compute $f^{\prime}(x)$.
c) Find all critical numbers of $f$. (These will depend on at least one of $a, h$, and $k$.)
d) Assume that $a<0$. Construct a first derivative sign chart for $f$.
e) Based on the information you've found above, classify the critical values of $f$ as maxima or minima.

## II. Describing families of functions in terms of parameters

Given a family of functions that depends on one or more parameters, your goal is to describe the key characteristics of the overall behavior of each member of the familiy in terms of those parameters. By finding the first and second derivatives and constructing first and second derivative sign charts (each of which may depend on one or more of the parameters), you can often make broad conclusions about how each member of the family will appear. The fundamental steps for this analysis are essentially identical to the work you did in Investigation 1, as is demonstrated through the following example.

Example 1. Consider the two-parameter family of functions given by $g(x)=a x e^{-b x}$, where $a$ and $b$ are positive real numbers. Fully describe the behavior of a typical member of the family in terms of $a$ and $b$, including the location of all critical numbers, where $g$ is increasing, decreasing, concave up, and concave down, and the long-term behavior of $g$.

Solution. You begin by computing $g^{\prime}(x)$. By the product rule,

$$
g^{\prime}(x)=a x \frac{d}{d x}\left[e^{-b x}\right]+e^{-b x} \frac{d}{d x}[a x]
$$

and thus by applying the chain rule and constant multiple rule, you find that

$$
g^{\prime}(x)=a x e^{-b x}(-b)+e^{-b x}(a)
$$

To find the critical numbers of $g$, you solve the equation $g^{\prime(x)}=0$. Here, it is especially helpful to factor $g^{\prime}$. You thus observe that setting the derivative equal to zero implies

$$
0=a x e^{-b x}(-b x+1)
$$

Since you are given that $a \neq 0$ and you know that $e^{-b x} \neq 0$ for all values of $x$, the only way the preceding equation can hold is when $-b x=0$. Solving for $x$, you find that $x=\frac{1}{b}$, and this is therefore the only critical number of g .

Now, recall that you have shown $g^{\prime}(x)=a x e^{-b x}(1-b x)$ and that the only critical number of g is $x=\frac{1}{b}$. This enables you to construct the first derivative sign chart for $g$, as seen below.


Notice that in $g^{\prime}(x)=a x e^{-b x}(1-b x)$, the term $a e^{-b x}$ is always positive, so the sign depends on the linear term $(1-b x)$, which is zero when $x=\frac{1}{b}$. Note that this line
has negative slope $(-b)$, so $(1-b x)$ is positive for $x<\frac{1}{b}$ and negative for $x>\frac{1}{b}$. Hence, you can not only conclude that $g$ is always increasing for $x<\frac{1}{b}$ and decreasing for $x>\frac{1}{b}$, but also that $g$ has a global maximum at $\left(\frac{1}{b}, g\left(\frac{1}{b}\right)\right)$ and no local minimum.

Turn next to analyzing the concavity of $g$. With $g^{\prime}(x)=a x e^{-b x}(-b)+e^{-b x}(a)$, you differentiate to find that

$$
g^{\prime \prime}(x)=-a b x e^{-b x}(-b)+e^{-b x}(-a b)+a e^{-b x}(-b)
$$

Combining like terms and factoring, you now have

$$
g^{\prime \prime}(x)=a b^{2} x e^{-b x}-2 a b e^{-b x}=a b e^{-b x}(b x-2)
$$

Similar to your work with the first derivative, observe that $a b e^{-b x}$ is always positive, and thus the sign of $g^{\prime \prime}$ depends on the sign of $(b x-2)$, which is zero when $x=\frac{2}{b}$. Since $(b x-2)$ represents a line with positive slope $b$, the value of $(b x-2)$ is negative for $x<\frac{2}{b}$ and positive for $x>\frac{2}{b}$, and thus the sign chart for $g^{\prime \prime}$ is given by the one shown below.


Thus, $g$ is concave down for all $x<\frac{2}{b}$ and concave up for all $x>\frac{2}{b}$.
Finally, you analyze the long term behavior of $g$ by considering two limits. First, note that

$$
\lim _{x \rightarrow \infty} g(x)=\lim _{x \rightarrow \infty} a x e^{-b x}=\lim _{x \rightarrow \infty} \frac{a x}{e^{b x}}
$$

Since this limit has indeterminate form $\frac{\infty}{\infty}$, you can apply L'Hopital's Rule and thus find that $\lim _{x \rightarrow \infty} g(x)=0$. In the other direction,

$$
\lim _{x \rightarrow-\infty} g(x)=\lim _{x \rightarrow-\infty} a x e^{-b x}=-\infty
$$

since $a x \rightarrow-\infty$ and $e^{-b x} \rightarrow-\infty$ as $x \rightarrow-\infty$. Hence, as you move left on its graph, $g$ decreases without bound, while as you move to the right, $g(x) \rightarrow 0$.

All of the above information now allows you to produce the graph of a typical member of this family of functions without using a graphing utility (and without choosing particular values for a and b ), as shown below.


Note that the value of $b$ controls the horizontal location of the global maximum and the inflection point, as neither depends on a. The value of a affects the vertical stretch of the graph. For example, the global maximum occurs at the point $\left(\frac{1}{b} \cdot g\left(\frac{1}{b}\right)\right)=\left(\frac{1}{b} \cdot \frac{1}{b} e^{-1}\right)$
so the larger the value of a, the greater the value of the global maximum.

The kind of work outlined in Example 1 can be replicated for other families of functions that depend on parameters. Normally you are most interested in determining all critical numbers, a first derivative sign chart, a second derivative sign chart, and some analysis of the limit of the function as $x \rightarrow \infty$. Throughout, strive to work with the parameters as arbitrary constants. If stuck, it is always possible to experiment with some particular values of the parameters present to reduce the algebraic complexity of your work.

Investigation 2: Consider the family of functions defined by $p(x)=x^{3}-a x$, where $a \neq 0$ is an arbitrary constant.
a) Find $p^{\prime}(x)$ and determine the critical numbers of $p$. How many critical numbers does $p$ have?
b) Construct a first derivative sign chart for $p$. What can you say about the overall behavior of $p$ if the constant $a$ is positive? Why? What if the constant a is negative? In each case, describe the relative extremes of $p$.
c) Find $p^{\prime \prime}(x)$ and construct a second derivative sign chart for p . What does this tell you about the concavity of $p$ ? What role does $a$ play in determining the concavity of $p$ ?
d) Without using a graphing utility, sketch and label typical graphs of $p(x)$ for the cases where $a>0$ and $a<0$. Label all inflection points and local extrema.
e) Finally, use a graphing utility to test your observations above by entering and plotting the function $p(x)=x^{3}-a x$ for at least four different values of $a$. Write several sentences to describe your overall conclusions about how the behavior of $p$ depends on $a$.

Investigation 3: Consider the two-parameter family of functions of the form $h(x)=$ $a\left(1-e^{-b x}\right)$, where $a$ and $b$ are positive real numbers.
a) Find the first derivative and the critical numbers of $h$. Use these to construct a first derivative sign chart and determine for which values of $x$ the function $h$ is increasing and decreasing.
b) Find the second derivative and build a second derivative sign chart. For which values of $x$ is a function in this family concave up? concave down?
c) What is the value of $\lim _{x \rightarrow \infty} a\left(1-e^{-b x}\right)$, and $\lim _{x \rightarrow-\infty} a\left(1-e^{-b x}\right)$ ?
d) How does changing the value of $b$ affect the shape of the curve?
e) Without using a graphing utility, sketch the graph of a typical member of this family. Write several sentences to describe the overall behavior of a typical function h and how this behavior depends on $a$ and $b$.

Investigation 4: Let $L(t)=\frac{A}{1+c e^{-k t}}$, where $A, c$, and $k$ are all positive real numbers.
a) Observe that you can equivalently write $L(t)=A\left(1+c e^{-k t}\right)^{-1}$. Find $L^{\prime}(t)$ and explain why $L$ has no critical numbers. Is $L$ always increasing or always decreasing? Why?
b) Given the fact that $L^{\prime \prime}(t)=A c k^{2} e^{-k t} \frac{c e^{-k t}-1}{\left(1+c e^{-k t}\right)^{3}}$
find all values of $t$ such that $L^{\prime \prime}(t)=0$ and hence construct a second derivative sign chart. For which values of $t$ is a function in this family concave up? concave down?
c) What is the value of $\lim _{t \rightarrow \infty} \frac{A}{1+c e^{-k t}}$ and $\lim _{t \rightarrow-\infty} \frac{A}{1+c e^{-k t}}$ ?
d) Find the value of $L(t)$ at the inflection point found in (b).
(e) Without using a graphing utility, sketch the graph of a typical member of this family. Write several sentences to describe the overall behavior of a typical function L and how this behavior depends on $A, c$, and $k$ critical number.
(f) Explain why it is reasonable to think that the function $L(t)$ models the growth of a population over time in a setting where the largest possible population the surrounding environment can support is $A$.

## III. Exercises

1. Consider the one-parameter family of functions given by $p(x)=x^{3}-a x^{2}$, where $a>0$.
a) Sketch a plot of a typical member of the family, using the fact that each is a cubic polynomial with a repeated zero at $x=0$ and another zero at $x=a$.
b) Find all critical numbers of $p$.
c) Compute $p^{\prime \prime}$ and find all values for which $p^{\prime \prime}(x)=0$. Hence construct a second derivative sign chart for $p$.
d) Describe how the location of the critical numbers and the inflection point of $p$ change as $a$ changes. That is, if the value of $a$ is increased, what happens to the critical numbers and inflection point?
2. Let $q(x)=\frac{e^{-x}}{x-c}$ be a one-parameter family of functions where $c>0$.
a) Explain why $q$ has a vertical asymptote at $x=c$.
b) Determine $\lim _{x \rightarrow \infty} q(x)$ and $\lim _{x \rightarrow-\infty} q(x)$.
(c) Compute $q^{\prime}(x)$ and find all critical numbers of q .
d) Construct a first derivative sign chart for $q$ and determine whether each critical number leads to a local minimum, local maximum, or neither for the function $q$.
e) Sketch a typical member of this family of functions with important behaviors clearly labeled.
3. Let $E(x)=\frac{(x-m)^{2}}{2 s^{2}}$, where m is any real number and s is a positive real number.
a) Compute $E^{\prime}(x)$ and hence find all critical numbers of $E$.
b) Construct a first derivative sign chart for $E$ and classify each critical number of the function as a local minimum, local maximum, or neither.
c) It can be shown that $E^{\prime \prime}(x)$ is given by the formula

$$
E^{\prime \prime(x)}=e^{-\frac{(x-m)^{2}}{2 s^{2}}}\left(\frac{(x-m)^{2}-s^{2}}{s^{4}}\right)
$$

Find all values of x for which $E^{\prime \prime(x)}=0$.
d) Determine $\lim _{x \rightarrow \infty} E(x)$ and $\lim _{x \rightarrow-\infty} E(x)$.
e) Construct a labeled graph of a typical function $E$ that clearly shows how important points on the graph of $y=E(x)$ depend on $m$ and $s$.

