### 2.27 Buried Treasures

Derivatives of Functions Given Implicitly

In all of your studies with derivatives to date, you have worked in a setting where you
 can express a formula for the function of interest explicitly in terms of $x$. But there are many interesting curves that are determined by an equation involving $x$ and $y$ for which it is impossible to solve for $y$ in terms of $x$. Perhaps the simplest and most natural of all such curves are circles.




Because of the circle's symmetry, for each $x$ value strictly between the endpoints of the horizontal diameter, there are two corresponding $y$-values. For
instance, in the figure above, you have labeled $A=(-3, \sqrt{7})$ and $B=(-3,-\sqrt{7})$, and these points demonstrate that the circle fails the vertical line test. Hence, it is impossible to represent the circle through a single function of the form $y=f(x)$. At the same time, portions of the circle can be represented explicitly as a function of $x$, such as the highlighted arc that is magnified in the center of figure. Moreover, it is evident that the circle is locally linear, so you ought to be able to find a tangent line to the curve at every point; thus, it makes sense to wonder if you can compute $\frac{d y}{d x}$ at any point on the circle, even though you cannot write $y$ explicitly as a function of $x$. Finally, you note that the right-hand curve above is called a lemniscate and is just one of many fascinating possibilities for implicitly given curves.

In working with implicit functions, you will often be interested in finding an equation for $\frac{d y}{d x}$ that tells you the slope of the tangent line to the curve at a point $(x, y)$. To do so, it will be necessary for you to work with $y$ while thinking of $y$ as a function of $x$, but without being able to write an explicit formula for $y$ in terms of $x$.

Investigation 1: Let $f$ be a differentiable function of $x$ (whose formula is not known) and recall that $\frac{d}{d x}[f(x)]$ and $f^{\prime}(x)$ are interchangeable notations. Determine each of the following derivatives of combinations of explicit functions of $x$, the unknown function $f$, and an arbitrary constant $c$.
a) $\frac{d}{d x}\left[x^{2}+f(x)\right]$
b) $\frac{d}{d x}\left[x^{2} f(x)\right]$
c) $\frac{d}{d x}\left[c+x+f(x)^{2}\right]$
d) $\frac{d}{d x}\left[f(x)^{2}\right]$
e) $\frac{d}{d x}[x(f(x)+f(c x)+c f(x)]$

## II. Implicit Differentiation

Because a circle is perhaps the simplest of all curves that cannot be represented explicitly as a single function of $x$, let's begin your exploration of implicit differentiation with the example of the circle given by $x^{2}+y^{2}=16$. It is visually apparent that this curve is locally linear, so it makes sense for you to want to find the slope of the tangent line to the curve at any point, and moreover to think that the curve is differentiable. The big question is: how do you find a formula for $\frac{d y}{d x}$, the slope of the tangent line to the circle at a given point on the circle?

By viewing $y$ as an implicit ${ }^{1}$ function of $x$, you essentially think of $y$ as some function whose formula $f(x)$ is unknown, but which you can differentiate. Just as $y$ represents an unknown formula, so too its derivative with respect to $x, \frac{d y}{d x}$, will be (at least temporarily) unknown.

Consider the equation $x^{2}+y^{2}=16$ and view $y$ as an unknown differentiable function of $x$. Differentiating both sides of the equation with respect to $x$, you have

$$
\frac{d}{d x}\left[x^{2}+y^{2}\right]=\frac{d}{d x}[16]
$$

On the right, the derivative of the constant 16 is 0 , and on the left you can apply the sum rule, so it follows that

$$
\frac{d}{d x}\left[x^{2}\right]+\frac{d}{d x}\left[y^{2}\right]=0
$$

Next, it is essential that you recognize the different roles being played by $x$ and $y$. Since $x$ is the independent variable, it is the variable with respect to which you are differentiating, and thus $\frac{d}{d x}\left[x^{2}\right]=2 x$. So far, that means

$$
2 x+\frac{d}{d x}\left[y^{2}\right]=0
$$

But $y$ is the dependent variable and $y$ is an implicit function of $x$. Therefore, when you want to compute $\frac{d}{d x}\left[y^{2}\right]$ it is identical to the situation in Investigation 1 , where you computed $\frac{d}{d x}\left[f(x)^{2}\right]$. In both situations, you have an unknown function being squared, and you seek the derivative of the result. This requires the chain rule, by which you find that $\frac{d}{d x}\left[y^{2}\right]=2 y \frac{d y}{d x}$. So, that gets you to

$$
2 x+2 y \frac{d y}{d x}=0
$$

Since your goal is to find an expression for $\frac{d y}{d x}$, just solve for it. Subtracting $2 x$ from both sides and dividing by $2 y$,

$$
\frac{d y}{d x}=-\frac{2 x}{2 y}=-\frac{x}{y}
$$

[^0]There are several important things to observe about the result that $\frac{d y}{d x}=-\frac{x}{y}$.
First, this expression for the derivative involves both $x$ and $y$. It makes sense that this should be the case, since for each value of $x$ between -4 and 4 , there are two corresponding points on the circle, and the slope of the tangent line is different at each of these points. Second, this formula is entirely consistent with your
 understanding of circles. If you consider the radius from the origin to the point $(a, b)$, the slope of this line segment is $m_{r}=\frac{b}{a}$. The tangent line to the circle at ( $a, b$ ) will be perpendicular to the radius, and thus have slope $m_{t}=-\frac{a}{b}$, as shown at left.

Finally, the slope of the tangent line is zero at $(0,4)$ and $(0,-4)$, and is undefined at $(-4,0)$ and $(4,0)$; all of these values are consistent with the formula $\frac{d y}{d x}=-\frac{x}{y}$.

Example 1. For the curve given implicitly by $x^{3}+y^{2}-2 x y=2$, shown below right, find the slope of the tangent line at $(-1,1)$.

Solution. Begin by differentiating the curve's equation implicitly. Taking the derivative of each side with respect to $x$,

$$
\frac{d y}{d x}\left[x^{3}+y^{2}-2 x y\right]=\frac{d y}{d x}[2]
$$

By the sum rule and the fact that the derivative of a constant is zero, you have

$$
\frac{d}{d x}\left[x^{3}\right]+\frac{d}{d x}\left[y^{2}\right]-\frac{d}{d x}[2 x y]=0
$$



The first uses the simple power rule, the second requires the chain rule (since $y$ is an implicit function of $x$ ), and the third necessitates the product rule (again since $y$ is a function of $x$ ). Applying these rules, you now find that

$$
\begin{gathered}
3 x^{2}+2 y \frac{d y}{d x}-\left[2 x \frac{d y}{d x}+2 y\right]=0 \\
3 x^{2}+2 y \frac{d y}{d x}-2 x \frac{d y}{d x}-2 y=0
\end{gathered}
$$

Remembering that your goal is to find an expression for $\frac{d y}{d x}$ so that you can determine the slope of a particular tangent line, you want to solve the preceding equation for $\frac{d y}{d x}$. To do so, isolate the variable (put the terms involving $\frac{d y}{d x}$ on one side of the equation) and then factor. So, subtracting $3 x^{2}-2 y$ from both sides, it follows that

$$
2 y \frac{d y}{d x}-2 x \frac{d y}{d x}=2 y-3 x^{2}
$$

Factoring the left side to isolate $\frac{d y}{d x}$, you have

$$
\frac{d y}{d x}(2 y-2 x)=2 y-3 x^{2}
$$

Finally, you divide both sides by $(2 y-2 x)$ and conclude that

$$
\frac{d y}{d x}=\frac{2 y-3 x^{2}}{2 y-2 x}
$$

Here again, the expression for $\frac{d y}{d x}$ depends on both x and y . To find the slope of the tangent line at $(-1,1)$, substitute this point in the formula for $\frac{d y}{d x}$, using the notation

$$
\left.\frac{d y}{d x}\right|_{(-1,1)}=\frac{2(1)-3(-1)^{2}}{2(1)-2(-1)}=-\frac{1}{4}
$$

This value matches a visual estimate of the slope of the tangent line shown in the graph on the previous page.

Example 2 shows that it is possible when differentiating implicitly to have multiple terms involving $\frac{d y}{d x}$. Regardless of the particular curve involved, your approach will be similar each time. After differentiating, expand so that each side of the equation is a sum of terms, some of which involve $\frac{d y}{d x}$. Next, addition and subtraction are used to
get all terms involving $\frac{d y}{d x}$ on one side of the equation, with all remaining terms on the other. Finally, factor to get a single instance of $\frac{d y}{d x}$, and then divide to solve for $\frac{d y}{d x}$. Note, too, that since $\frac{d y}{d x}$ is often a function of both $x$ and $y$, you use the notation

$$
\left.\frac{d y}{d x}\right|_{(a, b)}
$$

to denote the evaluation of $\frac{d y}{d x}$ at the point $(a, b)$. This is analogous to writing $f^{\prime}(a)$ when $f^{\prime}$ depends on a single variable.

Finally, there is a big difference between writing $\frac{d}{d x}$ and $\frac{d y}{d x}$. For example,

$$
\frac{d}{d x}\left[x^{2}+y^{2}\right]
$$

gives an instruction to take the derivative with respect to x of the quantity $x^{2}+y^{2}$, presumably where $y$ is a function of $x$. On the other hand,

$$
\frac{d y}{d x}\left(x^{2}+y^{2}\right.
$$

means the product of the derivative of $y$ with respect to $x$ with the quantity $x^{2}+y^{2}$. Understanding this notational subtlety is essential.

Investigation 2: Consider the curve defined by the equation $x=y^{5}-5 y^{3}+4 y$, whose graph is pictured below right.
a) Explain why it is not possible to express $y$ as an explicit function of $x$.
b) Use implicit differentiation to find a formula for $\frac{d y}{d x}$.
c) Use your result from part (b) to find an equation of the line tangent to the graph of $x=y^{5}-5 y^{3}+4 y$ at the point $(0,1)$.
d) Use your result from part (b) to determine all of the points at which the graph of $x=y^{5}-5 y^{3}+$
 $4 y$ has a vertical tangent line.

Two natural questions to ask about any curve involve where the tangent line can be vertical or horizontal. To be horizontal, the slope of the tangent line must be zero, while to be vertical, the slope must be undefined. It is typically the case when differentiating implicitly that the formula for $\frac{d y}{d x}$ is expressed as a quotient of functions of $x$ and $y$, say

$$
\frac{d y}{d x}=\frac{p(x . y)}{q(x, y)}
$$

Thus, you observe that the tangent line will be horizontal precisely when the numerator is zero and the denominator is nonzero, making the slope of the tangent line zero. Similarly, the tangent line will be vertical whenever $q(x, y)=0$ and $p(x, y) \neq 0$, making the slope undefined. If both x and y are involved in an equation such as $p(x, y)=0$, you try to solve for one of them in terms of the other, and then use the resulting condition in the original equation that defines the curve to find an equation in a single variable that you can solve to determine the point(s) that lie on the curve at which the condition holds. It is not always possible to execute the desired algebra due to the possibly complicated combinations of functions that often arise.

Investigation 3: Consider the curve defined by the equation
$y\left(y^{2}-1\right)(y-2)=x(x-1)(x-2)$, whose graph is pictured below right. Through implicit differentiation, it can be shown that

$$
\frac{d y}{d x}=\frac{(x-1)(x-2)+x(x-2)+x(x-1)}{\left(y^{2}-1\right)(y-2)+2 y^{2}(y-2)+y\left(y^{2}-1\right)}
$$

Use this fact to answer each of the following questions.
a) Determine all points $(x, y)$ at which the tangent line to the curve is horizontal. (Use technology appropriately to find the needed zeros of the relevant polynomial function.)
b) Determine all points $(x, y)$ at which the tangent line is vertical. (Use technology appropriately to find the needed zeros of the relevant polynomial function.)

c) Find the equation of the tangent line to the curve at one of the points where $x=1$.

Investigation 4: For each of the following curves, use implicit differentiation to find $d y / d x$ and determine the equation of the tangent line at the given point.
a) $x^{3}-y^{3}=6 x y,(-3,3)$
b) $\sin (y)+y=x^{3}+x,(0,0)$
c) $3 x e^{-x y}=y^{2},(0.619061,1)$

## III. Exercises

1. Consider the curve given by the equation $2 y^{3}+y^{2}-y^{5}=x^{4}-2 x^{3}+x^{2}$. Find all points at which the tangent line to the curve is horizontal or vertical. Be sure to use a graphing utility to plot this implicit curve and to visually check the results of algebraic reasoning that you use to determine where the tangent lines are horizontal and vertical.
2. For the curve given by the equation $\sin (x+y)+\cos (x-y)=1$, find the equation of the tangent line to the curve at the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$
3. Implicit differentiation enables you a different perspective from which to see why the rule $\frac{d}{d x}\left[a^{x}\right]=a^{x} \ln (a)$ holds, if you assume that $\frac{d}{d x}[\ln (a)]=1$. This exercise leads you through the key steps to do so.
a) Let $y=a^{x}$. Rewrite this equation using the natural logarithm function to write x in terms of $y$ (and the constant a).
b) Differentiate both sides of the equation you found in (a) with respect to $x$, keeping in mind that $y$ is implicitly a function of $x$.
(c) Solve the equation you found in (b) for $\frac{d y}{d x}$, and then use the definition of $y$ to write $\frac{d y}{d x}$ solely in terms of $x$. What have you found?

## IV. Assessment - Khan Academy

1. Complete the following online practice exercises in the seventh unit (Basic Differentiaion) of Khan Academy's AP Calculus AB course:
a. Review Worksheet (not graded): https://www.khanacademy.org/math/ap-calculus-ab/advanced-differentiation-ab/implicit-differentiation-advanced-ab/a/implicit-differentiation-review
b. https://www.khanacademy.org/math/ap-calculus-ab/advanced-differentiation-ab/implicit-differentiation-intro-ab/e/implicitdifferentiation

[^0]:    ${ }^{1}$ Essentially the idea of an implicit function is that it can be broken into pieces where each piece can be viewed as an explicit function of x , and the combination of those pieces constitutes the full implicit function. For the circle, you could choose to take the top half as one explicit function of x , and the bottom half as another.

