### 2.25 The Structure of Compositions

The Chain Rule

A composite function is one where the input variable x first passes through one function, and then the resulting output passes through another. For
 instance, consider the function

$$
C(x)=\sin \left(x^{2}\right),
$$

and observe that any input $x$ passes through a chain of functions. In particular, in the process that defines the function $C(x), x$ is first squared, and then the sine of the result is taken. Using an arrow diagram,

$$
x \rightarrow x^{2} \rightarrow \sin \left(x^{2}\right) .
$$

In terms of the elementary functions $f$ and $g$, notice that x is first input in the function g , and then the result is used as the input in $f$. Said differently, you can write

$$
C(x)=f(g(x))=\sin \left(x^{2}\right)
$$

and say that $C$ is the composition of $f$ and $g$. You will refer to $g$, the function that is first applied to $x$, as the innerfunction, while $f$, the function that is applied to the result, is the outer function.

The main question that you answer in the present section is: given a composite function $C(x)=f(g(x))$ that is built from differentiable functions $f$ and $g$, how do you compute $C^{\prime}(x)$ in terms of $f, g, f^{\prime}$, and $g^{\prime}$ ?

Before you discover this powerful new tool, and why it works, you need to be comfortable decomposing composite functions so that you can correctly identify the inner and outer functions

Investigation 1: For each composite function given below, state a formula for the inner function $g$ and the outer function $f$ so that the overall composite function can be written in the form $f(g(x))$. If the function is a sum, product, or quotient of basic functions, use the appropriate rule to determine its derivative.

| Function | Inner, $f$ | Outer, $g$ | $f^{\prime}$ | $g^{\prime}$ |
| :--- | :---: | :---: | :---: | :---: |
| 1a) $\sqrt{1+x}$ | $1+\mathrm{x}$ | $\sqrt{x}$ | 1 | $\frac{1}{2 \sqrt{x}}$ |
| b) $\sin (2 x)$ |  |  |  |  |
| c) $(x-1)^{3}$ |  |  |  |  |
| d) $(3 x+2)^{4}$ |  |  |  |  |
| e) $\tan \left(x^{2}\right)$ |  |  |  |  |
| h) $\sqrt{1+x^{2}}$ |  |  |  |  |
| j) $e^{\sin (x)}$ |  |  |  |  |

Investigation 2: Discovering the chain rule: Use the TI-Nspire CX CAS handheld to calculate the derivative of the composite functions above. (Alternative: use an online CAS program, such as the one at Symbolab.)

| Function, C | $C^{\prime}$ |
| :--- | :--- |
| 2a) $\sqrt{1+x}$ |  |
| b) $\sin (2 x)$ |  |
| c) $(x-1)^{3}$ |  |
| d) $(3 x+2)^{4}$ |  |
| e) $\tan \left(x^{2}\right)$ |  |
| h) $\sqrt{1+x^{2}}$ |  |
| j) $e^{\sin (x)}$ |  |

k) Based on these examples, can you see a pattern? Write out your conjecture by filling in the right side of the following equation with computations involving $f, g, f^{\prime}$, and $g^{\prime}$ ?:
$\frac{d}{d x}[f(g(x))]=\ldots$ (Chain Rule)

If you're confident in your formula, try the following problems and use your handheld to check your results.

Investigation 3: For each function given below, identify an inner function $g$ and outer function f to write the function in the form $=f(g(x))$. Then, determine $f^{\prime}(x), g^{\prime}(x)$, and $\frac{d}{d x}[f(g(x))]$, and finally apply the chain rule to determine the derivative of the given function.
a) $h(x)=\cos \left(x^{4}\right)$
b) $p(x)=\sqrt{\tan (x)}$
c) $s(x)=2^{\sin (x)}$
d) $z(x)=\cot ^{5}(x)$
e) $m(x)=\left(\sec (x)+e^{x}\right)^{9}$

## II. Using multiple rules simultaneously

The chain rule now joins the sum, constant multiple, product, and quotient rules in your collection of the different techniques for finding the derivative of a function through understanding its algebraic structure and the basic functions that constitute it. It takes substantial practice to get comfortable with navigating multiple rules in a single problem; using proper notation and taking a few extra steps can be particularly helpful as well. You demonstrate with an example and then provide further opportunity for practice in the following activity.

Example 1. Find a formula for the derivative of $h(t)=3^{t^{2}+2 t} \sec ^{4}(t)$.
Solution. First observe that the basic structure of $h$ is that it is the product of two functions: $h(t)=a(t) \cdot b(t)$, where $a(t)=3^{t^{2}+2 t}$ and $b(t)=\sec ^{4}(t)$. Therefore, you will need to use the product rule to differentiate h. When it comes time to differentiate $a$ and $b$ in their roles in the product rule, notice that each is a composite function, therefore the chain rule will be needed. Begin by working separately to compute $a^{\prime}(t)$ and $b^{\prime}(t)$.

Starting with $a(t)=f(g(t))=3^{t^{2}+2 t}$, and finding the derivatives of f and g ,

$$
\begin{array}{ll}
f(t)=3^{\mathrm{t}} & g(t)=t^{2}+2 t \\
f^{\prime}(t)=3^{\mathrm{t}} \ln (3) & g^{\prime}(t)=2 t+2
\end{array}
$$

Thus, by the chain rule, it follows that

$$
\begin{gathered}
\frac{d}{d t}[f(g(t))]=f^{\prime}\left(g(t) \cdot g^{\prime}(t)\right. \\
=3^{t^{2}+2 t} \ln (3)(2 t+2)
\end{gathered}
$$

Turning next to b , you write $b(t)=r(s(t))=\sec ^{4}(t)$ and find the derivatives of r and s. Doing so,

$$
\begin{array}{ll}
r(t)=t^{4} & s(t)=\sec (t) \\
r^{\prime}(t)=4 t^{3} & s^{\prime}(t)=\sec (t) \tan (t)
\end{array}
$$

By the chain rule, you now know that

$$
\begin{gathered}
\frac{d}{d t}[r(s(t))]=f^{\prime}\left(s(t) \cdot s^{\prime}(t)\right. \\
=4 \sec ^{3}(t) \sec (t) \tan (t) \\
=4 \sec ^{4}(t) \tan (t)
\end{gathered}
$$

Now you are finally ready to compute the derivative of the overall function $h$. Recalling that $h(t)=3^{t^{2}+2 t} \sec ^{4}(t)$, by the product rule you have

$$
h^{\prime}(t)=3^{t^{2}+2 t} \frac{d}{d t}\left[\sec ^{4}(t)\right]+4 \sec ^{4}(t) \frac{d}{d t}\left[3^{t^{2}+2 t}\right]
$$

From your work above with a and b, you know the derivatives of $3^{t 2+2 t}$ and $\sec ^{4}(t)$, and therefore

$$
h^{\prime}(t)=3^{t^{2}+2 t} 4 \sec ^{4}(t) \tan (t)+4 \sec ^{4}(t) 3^{t^{2}+2 t} \ln (3)(2 t+2)
$$

Investigation 4: For each of the following functions, find the function's derivative. State the rule(s) you use, label relevant derivatives appropriately, and be sure to clearly identify your overall answer.
a) $p(r)=4 \sqrt{r^{6}+2 e^{r}}$
b) $m(v)=\sin \left(v^{2}\right) \cos \left(v^{3}\right)$
c) $h(y)=\frac{\cos (10 y)}{e^{4 y}+1}$
d) $s(z)=2^{z^{2} \sec (z)}$
e) $c(x)=\sin \left(e^{x^{2}}\right)$

The chain rule now adds substantially to your ability to do different familiar problems that involve derivatives. Whether finding the equation of the tangent line to a curve, the instantaneous velocity of a moving particle, or the instantaneous rate of change of a certain quantity, if the function under consideration involves a composition of other functions, the chain rule is indispensable.

Investigation 5: Use known derivative rules, including the chain rule, as needed to answer each of the following questions.
a) Find an equation for the tangent line to the curve $y=\sqrt{e^{x}+3}$ at the point where $x=0$.
b) If $s(t)=\frac{1}{\left(t^{2}+1\right)^{3}}$ represents the position function of a particle moving horizontally along an axis at time $t$ (where $s$ is measured in inches and $t$ in seconds), find the particle's instantaneous velocity at $t=1$. Is the particle moving to the left or right at that instant?
c) At sea level, air pressure is 30 inches of mercury. At an altitude of $h$ feet above sea level, the air pressure, $P$, in inches of mercury, is given by the function
$P=30 e^{-0.0000323 \mathrm{~h}}$.
Compute $\frac{d P}{d h}$ and explain what this derivative function tells you about air pressure, including a discussion of the units on $\frac{d P}{d h}$. In addition, determine how fast the air pressure is changing for a pilot of a small plane passing through an altitude of 1000 feet.
d) Suppose that $f(x)$ and $g(x)$ are differentiable functions and that the following information about them is known:

| $x$ | $f(x)$ | $f^{\prime}(x)$ | $g(x)$ | $g^{\prime}(x)$ |
| :---: | :---: | :---: | :---: | :---: |
| -1 | 2 | -5 | -3 | 4 |
| 2 | -3 | 4 | -1 | 2 |

If $C(x)$ is a function given by the formula $f(g(x))$, determine $C^{\prime}(2)$. In addition, if $D(x)$ is the function $f(f(x))$, find $D^{\prime}(-1)$.

## III. The composite version of basic function rules

As you gain more experience with differentiating complicated functions, you will become more comfortable in the process of simply writing down the derivative without taking multiple steps. You demonstrate part of this perspective here by showing how you can find a composite rule that corresponds to two of your basic functions. For instance, you know that $\frac{d}{d x}[\sin (x)]=\cos (x)$. If you instead want to know

$$
\frac{d}{d x}[\sin (u(x))]
$$

where u is a differentiable function of $x$, then this requires the chain rule with the sine function as the outer function. Applying the chain rule,

$$
\frac{d}{d x}[\sin (u(x))]=\cos (u(x)) \cdot u^{\prime \prime}(x)
$$

Similarly, since $\frac{d}{d x}\left[a^{u(x)}\right]=a^{x} \ln (a) \cdot u^{\prime}(x)$, it follows by the chain rule that

$$
\frac{d}{d x}\left[a^{u(x)}\right]=a^{u(x)} \ln (a) \cdot u^{\prime}(x)
$$

In the process of getting comfortable with derivative rules, an excellent exercise is to write down a list of all basic functions whose derivatives are known, list those derivatives, and then write the corresponding chain rule for the composite version with the inner function being an unknown function $u(x)$ and the outer function being the known basic function. These versions of the chain rule are particularly simple when the inner function is linear, since the derivative of a linear function is a constant. For instance,

$$
\begin{gathered}
\frac{d}{d x}\left[(5 x+7)^{10}\right]=10(5 x+7)^{9} \cdot 5 \\
\frac{d}{d x}\left[(\tan (17 x)]=17 \sec ^{2}(17 x),\right. \text { and } \\
\frac{d}{d x}\left[\left(e^{-3 x}\right]=-3 e^{-3 x}\right.
\end{gathered}
$$

Chain Rule: If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$, the composite function C defined by $C(x)=f(g(x))$ is differentiable at $x$ and

$$
C^{\prime}(x)=f^{\prime}\left(g(x) \cdot g^{\prime}(x)\right.
$$

## IV. Exercises

1. Consider the basic functions $f(x)=x^{3}$ and $g(x)=\sin (x)$.
a) Let $h(x)=f(g(x))$ Find the exact instantaneous rate of change of $h$ at the point where $x=\frac{\pi}{4}$.
b) Which function is changing most rapidly at $x=0.25$ : $h(x)=f(g(x))$ or $r(x)=g(f(x))$ ? Why?
c) Let $h(x)=f(g(x))$ and $r(x)=g(f(x))$. Which of these functions has a derivative that is periodic? Why?
2. Let $u(x)$ be a differentiable function. For each of the following functions, determine the derivative. Each response will involve $u$ and/or $u$ '.
(a) $p(x)=e^{u(x)}$
(b) $q(x)=u\left(e^{x}\right)$
(c) $r(x)=\cot (u(x))$
(d) $s(x)=u(\cot (x))$
(e) $a(x)=u\left(x^{4}\right)$
(f) $b(x)=u^{4}(x)$
3. Let functions $p$ and $q$ be the piecewise linear functions given by their respective graphs at right. Use the graphs to answer the following questions.
(a) Let $C(x)=p(q(x))$. Determine $C^{\prime}(0)$ and $C^{\prime}(3)$.
(b) Find a value of x for which $C^{\prime}(x)$ does not exist. Explain your thinking.
(c) Let $\mathrm{Y}(\mathrm{x})=q(q(x))$ and $Z(x)=q(p(x))$.

Determine $Y^{\prime}(-2)$ and $Z^{\prime}(0)$.

4. If a spherical tank of radius 4 feet has $h$ feet of water present in the tank, then the volume of water in the tank is given by the formula

$$
V=\frac{\pi}{3} h^{2}(12-h)
$$

(a) At what instantaneous rate is the volume of water in the tank changing with respect to the height of the water at the instant $h=1$ ? What are the units on this quantity?
(b) Now suppose that the height of water in the tank is being regulated by an inflow and outflow (e.g., a faucet and a drain) so that the height of the water at time $t$ is given by the rule $h(t)=\sin (\pi t)+1$, where $t$ is measured in hours (and $h$ is still measured in feet). At what rate is the height of the water changing with respect to time at the instant $t=2$ ?
(c) Continuing under the assumptions in (b), at what instantaneous rate is the volume of water in the tank changing with respect to time at the instant $t=2$ ?
(d) What are the main differences between the rates found in (a) and (c)? Include a discussion of the relevant units.

## V. Practice - Khan Academy

1. Complete two online practice exercises in the Chain Rule unit of Khan Academy's AP Calculus AB course:
a. https://www.khanacademy.org/math/ap-calculus-ab/product-quotient-chain-rules-ab/chain-rule-ab/e/differentiate-compositefunctions
b. https://www.khanacademy.org/math/ap-calculus-ab/product-quotient-chain-rules-ab/chain-rule-ab/e/chain rule 1

## II. Optional

1. Challenge: https://www.khanacademy.org/math/ap-calculus-ab/product-quotient-chain-rules-ab/product-quotient-chain-rule-review-ab/e/combining-the-product-rule-and-chain-rule
