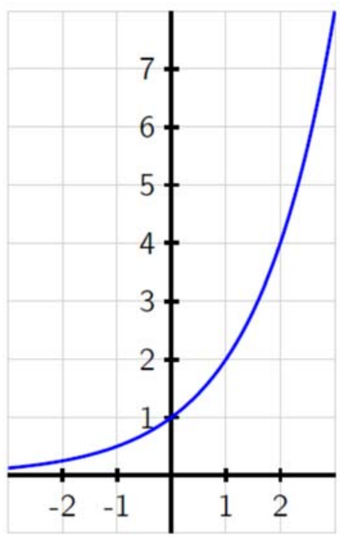
2.22 Periodic Behavior *Differentiating Basic Trig Functions*

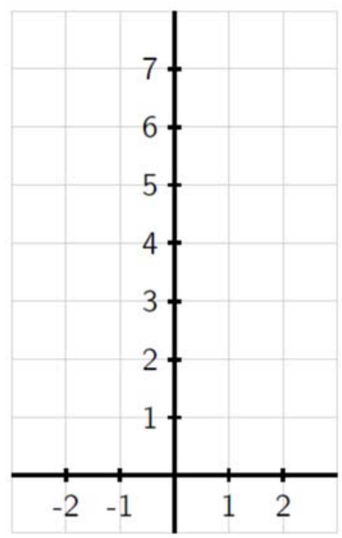
Trigonometry is one of the most practical pf mathematics, modeling many of the earth's periodic functions, including the periodic amount of sunlight in non-equatorial regions. Pictured above is the summer solstice in Antarctica!

Investigation 1: Consider the function $g(x) = 2^x$, which is graphed at left.

1a) At each of x = -2, -1, 0, 1, 2, use a straightedge to sketch an accurate tangent line to $y = g(x)$.

b) Use the provided grid to estimate the slope of the tangent line you drew at each point in (a).





c) Use the limit definition of the derivative to estimate $g'(0)$ by using small values of h, and compare the result to your visual estimate for the slope of the tangent line to $y = g(x)$ at $x = 0$ in (b).

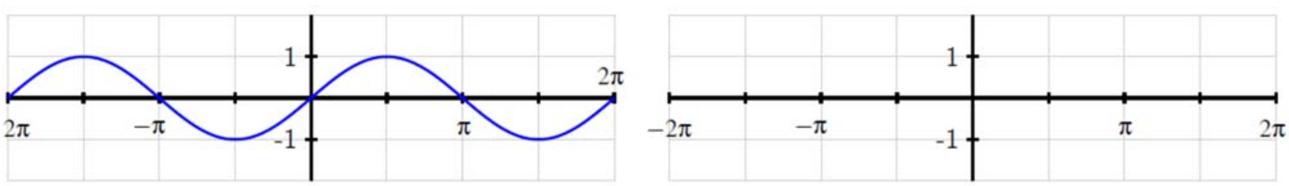
d) Based on your work in (a), (b), and (c), sketch an accurate graph of $y = g'(x)$ on the axes adjacent to the graph of $y = g(x)$.

e) Write at least one sentence that explains why it is reasonable to think that $g'(x) = c\, g(x)$, where c is a constant. In addition, calculate $\ln(2)$, and then discuss how this value, combined with your work above, reasonably suggests that $g'(x) = 2x \ln(2)$.

II. The sine and cosine functions

The sine and cosine functions are among the most important functions in all of mathematics. Sometimes called the circular functions due to their genesis in the unit circle, these periodic functions play a key role in modeling repeating phenomena such as the location of a point on a bicycle tire, the behavior of an oscillating mass attached to a spring, tidal elevations, and more. Like polynomial and exponential functions, the sine and cosine functions are considered basic functions, ones that are often used in the building of more complicated functions. As such, you would like to know formulas for $\frac{d}{dx}[\sin(x)]$ and $\frac{d}{dx}[\cos(x)]$, and the next two activities lead you to that end.

Investigation 2: Consider the function $f(x) = \sin(x)$, which is graphed below. Note carefully that the grid in the diagram does not have boxes that are 1×1, but rather approximately 1.57×1, as the horizontal scale of the grid is $\frac{\pi}{2}$ units per box.



2a) At each of $x = -2\pi. \frac{-3\pi}{2}, -\pi, \frac{-\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, use a straightedge to sketch an accurate tangent line to $y = f(x)$.

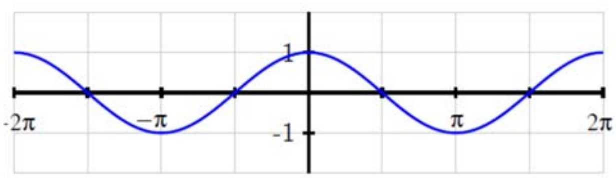
b) Use the provided grid to estimate the slope of the tangent line you drew at each point. Pay careful attention to the scale of the grid.

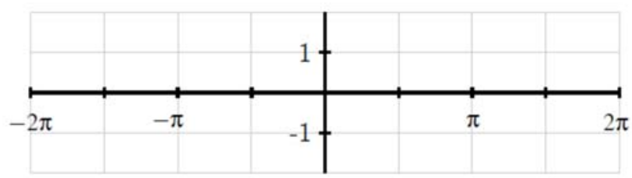
c) Use the limit definition of the derivative to estimate $f'(0)$ by using small values of h, and compare the result to your visual estimate for the slope of the tangent line to $y = f(x)$ at $x = 0$ in (b). Using periodicity, what does this result suggest about $f'(2\pi)$? about $f'(-2\pi)$?

d) Based on your work in (a), (b), and (c), sketch an accurate graph of $y = f'(x)$ on the axes adjacent to the graph of $y = f(x)$.

e) What familiar function do you think is the derivative of $f(x) = \sin(x)$?

Investigation 3: Consider the function $g(x) = \cos(x)$, which is graphed below. Note that the grid in the diagram does not have boxes that are 1×1, but rather approximately 1.57×1, as the horizontal scale of the grid is $\frac{\pi}{2}$ units per box.





a) At each of $x = -2\pi. \frac{-3\pi}{2}, -\pi, \frac{-\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi,,$ use a straightedge to sketch an accurate tangent line to $y = g(x)$.

b) Use the provided grid to estimate the slope of the tangent line you drew at each point. Again, note the scale of the axes and grid.

c) Use the limit definition of the derivative to estimate $g'(\frac{\pi}{2})$ by using small values of h, and compare the result to your visual estimate for the slope of the tangent line to $y = g(x)$ at $x = \frac{\pi}{2}$ in (b). Using periodicity, what does this result suggest about $g'(-\frac{3\pi}{2})$? Can symmetry on the graph help you estimate other slopes easily?

(d) Based on your work in (a), (b), and (c), sketch an accurate graph of $y = g'(x)$ on the axes adjacent to the graph of $y = g(x)$.

(e) What familiar function do you think is the derivative of $g(x) = \cos(x)$?

The following rules summarize the results of the two previous investigations.

Sine and Cosine Functions: For all real number x,

- $\frac{d}{dx}[\sin(x)] = \cos(x)$ and
- $\frac{d}{dx}[\cos(x)] = -\sin(x)$,

You have now added two additional functions to your library of basic functions whose derivatives you know: power functions, exponential functions, and the sine and cosine functions. The constant multiple and sum rules still hold, of course, and all of the inherent meaning of the derivative persists, regardless of the functions that are used to constitute a given choice of $f(x)$.

Investigation 4: Answer each of the following questions. Where a derivative is requested, be sure to label the derivative function with its name using proper notation.

a) Determine the derivative of $h(t) = 3\cos(t)$ - $4\sin(t)$.

b) Find the exact slope of the tangent line to $f(x) = 2x + \frac{\sin(x)}{2}$ at the point where $x = \frac{\pi}{2}$.

c) Find the equation of the tangent line to $g(x) = x^2 + 2\cos(x)$ at the point where $x = \frac{\pi}{2}$.

d) Determine the derivative of $p(z) = z^4 + 4^z + 4\cos(z) - \sin(\frac{\pi}{2})$.

e) The function $P(t) = 24 + 8\sin(t)$ represents the population of a animal that lives on a small island, where P is measured in hundreds and t is measured in decades since January 1, 2010. What is the instantaneous rate of change of P on January 1, 2030? What are the units of this quantity? Write a sentence in everyday language that explains how the population is behaving at this point in time.

III. Exercises

1. Suppose that $V(t) = 24 \cdot 1.07^t + 6\sin(t)$ represents the value of a person's investment portfolio in thousands of dollars in year t, where $t = 0$ corresponds to January 1, 2010.

   a) At what instantaneous rate is the portfolio's value changing on January 1, 2012? Include units on your answer.

   b) Determine the value of V"(2) . What are the units on this quantity and what does it tell you about how the portfolio's value is changing?

   c) On the interval $0 \leq t \leq 20$, graph the function $V(t) = 24 \cdot 1.07^t + 6\sin(t)$ and describe its behavior in the context of the problem. Then, compare the graphs of the functions $A(t) = 24 \cdot 1.07^t$ and V (t) = 24 · 1.07t + 6 sin(t), as well as the graphs of their derivatives A'(t) and V'(t). What is the impact of the term 6 sin(t) on the behavior of the function $V(t)$?

2. Let $f(x) = 3\cos(x) - 2\sin(x) + 6$.

   a) Determine the exact slope of the tangent line to $y = f(x)$ at the point where $a = \frac{\pi}{4}$.

   b) Determine the tangent line approximation to $y = f(x)$ at the point where $a = \pi$.

   c) At the point where $a = \frac{\pi}{2}$, is f increasing, decreasing, or neither?

d) At the point where $a = \frac{3\pi}{2}$, does the tangent line to $y = f(x)$ lie above the curve, below the curve, or neither? How can you answer this question without even graphing the function or the tangent line?

3. In this exercise, explore how the limit definition of the derivative more formally shows that $\frac{d}{dx}[\sin(x)] = \cos(x)$. Letting $f(x) = \sin x$, note that the limit definition of the derivative tells you that

$$f'(x) = \lim_{h \to 0} \frac{sin(x+h) - sin(x)}{h}$$

a) Recall the trigonometric identity for the sine of a sum of angles α and β: $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$. Use this identity and some algebra to show that

$$f'(x) = \lim_{h \to 0} \frac{sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h}$$

b) Next, note that as h changes, x remains constant. Explain why it therefore makes sense to say that

$$f'(x) = \sin(x) \lim_{h \to 0} \frac{(\cos(h) - 1)}{h} + +\cos(x) \cdot \lim_{h \to 0} \frac{\sin(h)}{h}$$

c) Finally, use small values of h to estimate the values of the two limits in (c):

d) What do your results in (c) thus tell you about $f'(x)$?

e) By emulating the steps taken above, use the limit definition of the derivative to argue convincingly that $\frac{d}{dx}[\cos(x)] = -\sin(x)$.

IV. Practice – Khan Academy

1. Complete the following online practice exercises in the sixth unit (Basic Differentiation) of Khan Academy's AP Calculus AB course: https://www.khanacademy.org/math/ap-calculus-ab/basic-differentiation-ab?t=practice
2. Optional: none