### 2.1F On a Tangent

The Tangent Line<br>Approximation

Among all functions, linear functions
 are simplest. One of the powerful consequences of a function $y=f(x)$ being differentiable at a point $(a, f(a))$ is that, up close, the function $y=f(x)$ is locally linear and looks like its tangent line at that point. In certain circumstances, this allows you to approximate the original function $f$ with a simpler function $L$ that is linear: this can be advantageous when you have limited information about $f$ or when $f$ is computationally or algebraically complicated.

It is essential to recall that when $f$ is differentiable at $x=a$, the value of $f^{\prime}(a)$ provides the slope of the tangent line to $\mathrm{y}=\mathrm{f}(\mathrm{x})$ at the point $(a, f(a)$. By knowing both a point on the line and the slope of the line you are thus able to find the equation of the tangent line. Investigation 1 will refresh these concepts through a key example and set the stage for further study.

Investigation 1: Consider the function $y=g(x)=-x^{2}+3 x+2$.
1a) Use the limit definition of the derivative to compute a formula for $y=g^{\prime}(x)$.
b) Determine the slope of the tangent line to $y=g(x)$ at the value $x=2$.
c) Compute $g(2)$.
d) Find an equation for the tangent line to $y=g(x)$ at the point $(2, g(2))$. Write your result in point-slope form ${ }^{1}$.

[^0]
e) On the axes provided above, sketch an accurate, labeled graph of $y=g(x)$ along with its tangent line at the point $(2, g(2))$.

## II. The tangent line

Given a function f that is differentiable at $x=a$, you can determine the slope of the tangent line to $y=f(x)$ at $(a, f(a))$ by computing $f^{\prime}(a)$. The resulting tangent line through $(a, f(a))$ with slope $m=f^{\prime}(a)$ has its equation in point-slope form given by

$$
y-f(a)=f^{\prime}(a)(x-a),
$$

which you can also express as $y=f^{\prime}(a)(x-a)+f(a)$. Note well: there is a major difference between $f(a)$ and $f(x)$ in this context. The former is a constant that results from using the given fixed value of a, while the latter is the general expression for the rule that defines the function. The same is true for $f^{\prime}(a)$ and $f^{\prime}(x)$ : you must carefully distinguish between these expressions. Each time you find the tangent line, you need to evaluate the function and its derivative at a fixed $a$-value.

In the figure below, there is a labeled plot of the graph of a function $f$ and its tangent line at the point $(a, f(a))$. Notice how when you zoom in you see the local linearity of $f$ more clearly highlighted as the function and its tangent line are nearly indistinguishable up close.


## III. The local linearization

A slight change in perspective and notation will enable us to be more precise in discussing how the tangent line to $y=f(x)$ at $(a, f(a))$ approximates $f$ near $x=a$. Taking the equation for the tangent line and solving for $y$, you observe that the tangent line is given by

$$
y=f^{\prime}(a)(x-a)+f(a)
$$

and that this line is itself a function of x . Replacing the variable y with the expression $\mathrm{L}(\mathrm{x})$, you call

$$
L(x)=f^{\prime}(a)(x-a)+f(a)
$$

the local linearization of $f$ at the point $(a, f(a))$. In this notation, it is particularly important to observe that $L(x)$ is nothing more than a new name for the tangent line, and that for $x$ close to $a$, you have that $f(x) \approx L(x)$.

Say, for example, that you know that a function $y=f(x)$ has its tangent line approximation given by $L(x)=3-2(x-1)$ at the point $(1,3)$, but you do not know anything else about the function $f$. If you are interested in estimating a value of $f(x)$ for $x$ near 1 , such as $f(1.2)$, you can use the fact that $f(1.2) \approx L(1.2)$ and hence

$$
f(1.2) \approx L(1.2)=3-2(1.2-1)=3-2(0.2)=2.6
$$

Again, much of the new perspective here is only in notation since $y=L(x)$ is simply a new name for the tangent line function. In light of this new notation and your observations above, you note that since $L(x)=f(a)+f^{\prime}(a)(x-a)$ and $L(x) \approx f(x)$ for $x$ near $a$, it also follows that you can write

$$
f(x) \approx f(a)+f^{\prime}(a)(x-a) \text { for } x \text { near } a
$$

Investigation 2: Suppose it is known that for a given differentiable function $y=g(x)$, its local linearization at the point where $a=-1$ is given by $L(x)=-2+3(x+1)$.

2a) Compute the values of $L(-1)$ and $L^{\prime}(-1)$.
b) What must be the values of $g(-1)$ and $g^{\prime}(-1)$ ? Why?
c) Do you expect the value of $g(-1.03)$ to be greater than or less than the value of $g(-1)$ ? Why?
d) Use the local linearization to estimate the value of $g(-1.03)$.
e) Suppose that you also know that $g^{\prime \prime}(? 1)=2$. What does this tell you about the graph of $y=g(x)$ at $\mathrm{a}=-1$ ?
f) For $x$ near -1 , sketch the graph of the local linearization $y=L(x)$ as well as a possible graph of $y=g(x)$ on the axes provided below.


As you saw in the example provided by Investigation 2, the local linearization $y=L(x)$ is a linear function that shares two important values with the function $y=f(x)$ that it is derived from. In particular, observe that since $L(x)=f(a)+f^{\prime}(a)(x-a)$, it follows that $L(a)=f(a)$. In addition, since $L$ is a linear function, its derivative is its slope.
Hence, $L^{\prime}(x)=f^{\prime}(a)$ for every value of $x$, and specifically $L^{\prime}(a)=f^{\prime}(a)$. Therefore, you see that $L$ is a linear function that has both the same value and the same slope as the function $f$ at the point $(a, f(a))$.

In situations where you know the linear approximation $y=L(x)$, you therefore know the original function's value and slope at the point of tangency. What remains unknown, however, is the shape of the function $f$ at the point of tangency. There are essentially four possibilities, as enumerated below.



These stem from the fact that there are three options for the value of the second derivative: either $f^{\prime \prime}(a)<0, f^{\prime \prime}(a)=0$, or $f^{\prime \prime}(a)>0$. If $f^{\prime \prime}(a)>0$, then you know the graph of $f$ is concave up, and you see the first possibility on the left, where the tangent line lies entirely below the curve. If $f$ " $(a)<0$, then you find yourselves in the second situation (from left) where $f$ is concave down and the tangent line lies above the curve. In the situation where $f^{\prime \prime}(a)=0$ and $f$ "changes sign at $\mathrm{x}=\mathrm{a}$, the concavity of the graph will change, and you will see either the third or fourth option ${ }^{2}$. A fifth option (that is not very interesting) can occur, which is where the function $f$ is linear, and so $f(x)=L(x)$ for all values of $x$.

The plots in the graphs above highlight yet another important aspect you can learn from the concavity of the graph near the point of tangency: whether the tangent line lies above or below the curve itself. This is key because it tells us whether or not the tangent line approximation's values will be too large or too small in comparison to the true value of $f$. For instance, in the first situation in the leftmost plot where $f^{\prime \prime}(a)>0$, since the tangent line falls below the curve, you know that $L(x) \leq f(x)$ for all values of $x$ near $a$.

[^1]Investigation 3: This activity concerns a function $f(x)$ about which the following information is known:

- $f$ is a differentiable function defined at every real number $x$
- $f(2)=-1$
- $y=f^{\prime}(x)$ has its graph given below




Your task is to determine as much information as possible about $f$ (especially near the value $a=2$ ) by responding to the questions below.
a) Find a formula for the tangent line approximation, $L(x)$, to $f$ at the point $(2,-1)$.
b) Use the tangent line approximation to estimate the value of $f(2.07$ _ . Show your work carefully and clearly.
c) Sketch a graph of $y=f^{\prime \prime}(x)$ on the right-hand grid above; label it appropriately.
d) Is the slope of the tangent line to $y=f(x)$ increasing, decreasing, or neither when $x=2$ ? Explain.
e) Sketch a possible graph of $y=f(x)$ near $x=2$ on the left-hand grid above. Include a sketch of $y=L(x)$ (found in part (a)). Explain how you know the graph of $y=f(x)$ looks like you have drawn it.
f) Does your estimate in (b) over- or under-estimate the true value of $f(2.07)$ ? Why?

The idea that a differentiable function looks linear and can be well-approximated by a linear function is an important one that finds wide application in calculus. For example, by approximating a function with its local linearization, it is possible to develop an effective
algorithm to estimate the zeroes of a function. Local linearity also helps us to make further sense of certain challenging limits. For instance, you have seen that a limit such as

$$
\lim _{x \rightarrow 0} \frac{\sin (x)}{x}
$$

is indeterminate because both its numerator and denominator tend to 0 . While there is no algebra that you can do to simplify $\frac{\sin (x)}{x}$, it is straightforward to show that the linearization of $f(x)=\sin (x)$ at the point $(0,0)$ is given by $L(x)=x$. Hence, for values of $x$ near 0 , $\sin (x) \approx x$. As such, for values of $x$ near 0 ,

$$
\frac{\sin (x)}{x} \approx \frac{x}{x}=1
$$

which makes plausible the fact that

$$
\lim _{x \rightarrow 0} \frac{\sin (x)}{x}
$$

## III Exercises

1. A certain function $y=p(x)$ has its local linearization at $a=3$ given by $L(x)=-2 x+5$.
a) What are the values of $p(3)$ and $p^{\prime}(3)$ ? Why?
b) Estimate the value of $p(2.79)$.
c) Suppose that $p^{\prime \prime}(3)=0$ and you know that $p "(x)<0$ for $x<3$. Is your estimate in (b) too large or too small?

Suppose that $p "(x)>0$ for $x>3$. Use this fact and the additional information above to sketch an accurate graph of $y=p(x)$ near $x=3$. Include a sketch of $y=L(x)$ in your work.
2. A potato is placed in an oven, and the potato's temperature $F$ (in degrees Fahrenheit) at various points in time is taken and recorded in the following table. Time $t$ is measured in minutes.

| $t$ | $F(t)$ |
| :---: | :---: |
| 0 | 70 |
| 15 | 180.5 |
| 30 | 251 |
| 45 | 296 |
| 60 | 324.5 |
| 75 | 342.8 |
| 90 | 354.5 |

a) Use a central difference to estimate $F^{\prime}(60)$. Use this estimate as needed in subsequent questions.
b) Find the local linearization $y=L(t)$ to the function $y=F(t)$ at the point where $a=$ 60.
c) Determine an estimate for $F(63)$ by employing the local linearization.
d) Do you think your estimate in (c) is too large or too small? Why?
3. An object moving along a straight line path has a differentiable position function $y=s(t)$; $s(t)$ measures the object's position relative to the origin at time $t$. It is known that at time $t=9$ seconds, the object's position is $s(9)=4$ feet (i.e., 4 feet to the right of the origin). Furthermore, the object's instantaneous velocity at $t=9$ is 1.2 feet per second, and its acceleration at the same instant is 0.08 feet per second per second.
a) Use local linearity to estimate the position of the object at $t=9.34$.
b) Is your estimate likely too large or too small? Why?
c) In everyday language, describe the behavior of the moving object at $t=9$. Is it moving toward the origin or away from it? Is its velocity increasing or decreasing?
4. For a certain function $f$, the derivative is known to be $f^{\prime \prime}(x)=(x-1) e^{-x^{2}}$. Note that you do not know a formula for $y=f(x)$.
a) At what $x$-value(s) is $f^{\prime}(x)=0$ ? Justify your answer algebraically, but include a graph of $f$ ' to support your conclusion.
b) Reasoning graphically, for what intervals of $x$-values is $f^{\prime \prime}(x)>0$ ? What does this tell you about the behavior of the original function $f$ ? Explain.
c) Assuming that $f(2)=3$, estimate the value of $f(1.88)$ by finding and using the tangent line approximation to f at $x=2$. Is your estimate larger or smaller than the true value of $f(1.88)$ ? Justify your answer.
IV. Practice - Khan Academy

1. Complete the following online practice exercises in the Derivative Applications unit of Khan Academy's AP Calculus AB course:
a. https://www.khanacademy.org/math/ap-calculus-ab/derivative-applications-ab/linear-approximation-ab/e/local-linearization
II. Optional
2. None

[^0]:    ${ }^{1}$ Recall that a line with slope $m$ that passes through $\left(x_{1}, y_{1}\right)$ has equation $y-y_{1}=m\left(x-x_{1}\right)$, and this is the point-slope form of the equation.

[^1]:    ${ }^{2}$ It is possible to have $f^{\prime \prime}(a)=0$ and have $f$ "not change sign at $x=a$, in which case the graph will look like one of the first two options.

