### 2.15 Making Connections

Limits, Continuity and Differentiability



Earlier, you learned about how the concept of limits can be used to study the trend of a function near a fixed input value. As you study such trends, you are fundamentally interested in knowing how well-behaved the function is at the given point, say $x=a$. In this lesson, you will expand your perspective and develop language and understanding to quantify how the function acts and how its value changes near a particular point. Beyond thinking about whether or not the function has a limit $L$ at $x=a$, you will also consider the value of the function $f(a)$ and how this value is related to $\lim _{x \rightarrow a} f(x)$, as well as whether or not the function has a derivative $f^{\prime}(a)$ at the point of interest. Throughout, you will build on and formalize ideas that you have encountered in several settings.

Investigation 1: A function $f$ defined on $-4<x<4$ is given by the graph below. Use the graph to answer each of the following questions. Note: to the right of $x=2$, the graph of $f$ is exhibiting infinite oscillatory behavior similar to the function $\sin \left(\frac{\pi}{4}\right)$ that you encountered earlier.

1a) For each of the values $a=-3,-2,-1,0,1,2,3$, determine whether $\lim _{x \rightarrow a} f(x)$ exists. If the function has a limit $L$ at a given point, state the value of the limit using the notation $\lim _{x \rightarrow a} f(x)$. If the function does not have a limit at a given point, write a sentence to explain why.

b) For each of the values of a from part (a) where $f$ has a limit, determine the value of $f(a)$ at each such point. In addition, for each such a value, does $f(a)$ have the same value as $\lim _{x \rightarrow a} f(x)$ ?
c) For each of the values $a=-3,-2,-1,0,1,2,3$, determine whether or not $f^{\prime}(a)$ exists. In particular, based on the given graph, ask yourself if it is reasonable to say that $f$ has a tangent line at $(a, f(a))$ for each of the given $a$-values. If so, visually estimate the slope of the tangent line to find the value of $f^{\prime}(a)$.

## II. Having a limit at a point

There are two behaviors that a function can exhibit at a point where it fails to have a limit. In the figure below, you see a function $f$ whose graph shows a jump at a $=1$. In particular, if you let $x$ approach 1 from the left side, the value of $f$ approaches 2 , while if you let $x$ go to 1
 from the right, the value of $f$ tends to 3 . Because the value of $f$ does not approach a single number as $x$ gets arbitrarily close to 1 from both sides, you know that $f$ does not have a limit at $a=1$.

Since $f$ does approach a single value on each side of $a=1$, you can introduce the notion of left and right (or onesided) limits. You say that $f$ has limit $L_{1}$ as $x$ approaches a from the left and write

$$
\lim _{x \rightarrow a^{-}} f(x)=L_{1}
$$

provided that you can make the value of $f(x)$ as close to $L_{1}$ as you like by taking $x$ sufficiently close to $a$ while always having $x<a$. In this case, you call $L_{1}$ the left-hand limit of $f$ as $x$ approaches $a$. Similarly, you say $L_{2}$ is the right-hand limit of $f$ as $x$ approaches $a$ and write

$$
\lim _{x \rightarrow a^{+}} f(x)=L_{2}
$$

provided that you can make the value of $f(x)$ as close to $L_{2}$ as you like by taking $x$ sufficiently close to $a$ while always having $x>a$. In the graph of the function $f$, you see that

$$
\lim _{x \rightarrow 1^{-}} f(x)=2 \quad \text { and } \quad \lim _{x \rightarrow 1^{+}} f(x)=3
$$

and precisely because the left and right limits are not equal, the overall limit of $f$ as $x \rightarrow 1$ fails to exist.

For the function $g$ pictured at the right, the function fails to have a limit at $a=1$ for a different reason. While the function does not have a jump in its graph at $a=1$, it is still not the case that $g$ approaches a single value as $x$ approaches 1 . In particular, due to the infinitely oscillating behavior of $g$ to the right of $a=1$, you say that the righthand limit of g as $x \rightarrow 1^{+}$does not exist, and thus
$\lim _{x \rightarrow 1} g(x)$ does not exist.


To summarize, anytime either a left- or right-hand limit fails to exist or the left- and righthand limits are not equal to each other, the overall limit will not exist. Said differently,

A function $f$ has a limit $L$ if and only if

$$
\lim _{x \rightarrow a^{+}} f(x)=L=\lim _{x \rightarrow a^{-}} f(x)
$$

That is, a function has a limit at $x=a$ if and only if both the left- and right-hand limits at $x=a$ exist and share the same value.

In Investigation 1, the function f given only fails to have a limit at two values: at $a=-2$ (where the left- and right-hand limits are 2 and -1 , respectively) and at $x=2$, where $\lim _{x \rightarrow 2^{+}} f(x)$ does not exist). Note well that even at values like $a=1$ and $a=0$ where there are holes in the graph, the limit still exists.

Investigation 2: Consider a function that is piecewise-defined by the formula

$$
f(x)=\left\{\begin{array}{cc}
3(x+2)+2 & \text { for }-3<x-2 \\
\frac{2}{3}(x+2)+1 & \text { for }-2 \leq x<-1 \\
\frac{2}{3}(x+2)+1 & \text { for }-1<x<-1 \\
2 & \\
4-x & \text { for } x=1
\end{array}\right.
$$

Use the given formula to answer the following questions.

2a) For each of the values $a=-2,-1,0,1,2$, compute $f(a)$.
b) For each of the values $a=-2,-1,0,1,2$, determine $\lim _{x \rightarrow a^{-}} f(x)$ and $\lim _{x \rightarrow a^{+}} f(x)$.

c) For each of the values $a=-2,-1,0,1,2$, determine $\lim _{x \rightarrow a} f(x)$. If the limit fails to exist, explain why by discussing the left- and right-hand limits at the relevant $a$-value.
d) For which values of $a$ is the following statement true?

$$
\lim _{x \rightarrow a} f(x) \neq f(a)
$$

e) On the axes provided above, sketch an accurate, labeled graph of $y=f(x)$. Be sure to carefully use open circles $\left({ }^{\circ}\right)$ and filled circles $(\bullet)$ to represent key points on the graph, as dictated by the piecewise formula.

## III. Being continuous at a point

Intuitively, a function is continuous if you can draw it without ever lifting your pencil from the page. Alternatively, you might say that the graph of a continuous function has no jumps or holes in it. Consider the three specific situations in the figure below where all three functions have a limit at $a=1$, and then work to make the idea of continuity more precise.




In the first graph at left, notice that $f(1)$ is not defined, which leads to the resulting hole in the graph of $f$ at $a=1$. You will say that $f$ is not continuous at $a=1$.

For the middle function $g$, observe that while $\lim _{x \rightarrow 1} g(x)=3$, the value of $g(1)=2$, and thus the limit does not equal the function value. Here, too, you will say that $g$ is not continuous, even though the function is defined at $a=1$.

Finally, the function $h$ appears to be the most well-behaved of all three, since at $a=1$ its limit and its function value agree. That is, $\lim _{x \rightarrow 1} h(x)=h(x)$. With no hole or jump in the graph of $h$ at $a=1$, you desire to say that $h$ is continuous there.

More formally, you make the following definition.

Definition: A function $f$ is continuous at $x=a$ provided that
a) $f$ has a limit as $x \rightarrow a$,
b) $f$ is defined at $x=a$, and
c) $\lim _{x \rightarrow a} f(x)=f(a)$

Conditions (a) and (b) are technically contained implicitly in (c), but they are stated here
explicitly to emphasize their individual importance.
If a function is continuous at every point in its domain, you say the function is "continuous." Thus, continuous functions are particularly nice: to evaluate the limit of a continuous function at a point, all you need to do is evaluate the function.

For example, consider $p(x)=x^{2}-2 x+3$. It can be proved that every polynomial is a continuous function at every real number, and thus if you would like to know $\lim _{x \rightarrow 2} p(x)$, you simply compute

$$
\lim _{x \rightarrow 2}\left(x^{2}-2 x+3\right)=2^{2}-2 \cdot 2+3=3
$$

This route of substituting an input value to evaluate a limit works anytime you know the function being considered is continuous. Besides polynomial functions, all exponential functions and the sine and cosine functions are continuous at every point, as are many other familiar functions and combinations thereof.

Investigation 3: This activity builds on your work in Investigation 1, using the same function $f$ as given by the graph that is repeated at right.

3a) At which values of a does $\lim _{x \rightarrow a} f(x)$ not exist?
b) At which values of a is $f(a)$ not defined?
c) At which values of a does $f$ have a limit, but
 $\lim _{x \rightarrow a} f(x) \neq f(a) ?$
d) State all values of $a$ for which $f$ is not continuous at $x=a$.
e) Which condition is stronger, and hence implies the other: $f$ has a limit at $x=a$ or $f$ is continuous at $x=a$ ? Explain, and hence complete the following sentence: "If $f$
$\qquad$ at $x=a$, then $f$ $\qquad$ at $x=a$," where you complete the blanks with has a limit and is continuous, using each phrase once.

## IV. Being differentiable at a point

Recall that a function $f$ is said to be differentiable at $x=a$ whenever $f^{\prime}(a)$ exists. Moreover, for $f^{\prime}(a)$ to exist, you know that the function $y=f(x)$ must have a tangent line at the point ( $a, f(a)$ ), since $f^{\prime}(a)$ is precisely the slope of this line. In order to even ask if f has a tangent line at $(a, f(a))$, it is necessary that $f$ be continuous at $x=a$ : if $f$ fails to have a limit at $x=a$, if $\mathrm{f}(\mathrm{a})$ is not defined, or if $\mathrm{f}(\mathrm{a})$ does not equal the value of $\lim _{x \rightarrow a} f(x)$, then it doesn't even make sense to talk about a tangent line to the curve at this point.




Indeed, it can be proved formally that if a function $f$ is differentiable at $x=a$, then it must be continuous at $x=a$. So, if $f$ is not continuous at $x=a$, then it is automatically the case that $f$ is not differentiable there. For example, in the graphs above, from your early discussion of continuity, both $f$ and $g$ fail to be differentiable at $x=1$ because neither function is continuous at $x=1$.

Function $h$ is the only graph that is differentiable at $x=1$.
But can a function fail to be differentiable at a point where the function is continuous?


In the figure above you revisit the situation where a function has a sharp corner at a point, something you encountered several times in your earlier work with Limits. For the pictured function f , observe that $f$ is clearly continuous at $a=1$, since $\lim _{x \rightarrow 1} f(1)=f(1)$.

But the function $f$ is not differentiable at $a=1$ because $f^{\prime}(1)$ fails to exist. One way to see this is to observe that $f^{\prime}(x)=1$ for every value of $x$ that is less than 1 , while $f^{\prime}(x)=+1$ for every value of $x$ that is greater than 1 . That makes it seem that either +1 or 1 would be equally good candidates for the value of the derivative at $x=1$. Alternately, you could use the limit definition of the derivative to attempt to compute $f^{\prime}(1)$, and discover that the derivative does not exist. Finally, you can also see visually that the function $f$ does not have a tangent line. When you zoom in on 1,1 on the graph of $f$, no matter how closely you examine the function, it will always look like a " V ", and never like a single line, which tells you there is no possibility for a tangent line there.

To make a more general observation, if a function does have a tangent line at a given point, when you zoom in on the point of tangency, the function and the tangent line should appear essentially indistinguishable. Conversely, if you have a function such that when you zoom in on a point the function looks like a single straight line, then the function should have a tangent line there, and thus be differentiable. Hence, a function that is differentiable at $x=a$ will, up close, look more and more like its tangent line at ( $a, f(a)$ ), and thus you say that a function is differentiable at $x=a$ is locally linear.

Investigation 4: In this activity, you explore two different functions and classify the points at which each is not differentiable. Let $g$ be the function given by the rule $g(x)=x$, and let $f$ be the function that you have previously explored in Investigation 1, whose graph is given again on the next page.
a) Reasoning visually, explain why $g$ is differentiable at every point $x$ such that
$x \neq 0$.
b) Use the limit definition of the derivative to show that $g^{\prime}(0)=g^{\prime}(0)=\lim _{h \rightarrow 0} \frac{|h|}{h}$.
c) Explain why $g^{\prime}(0)$ fails to exist by using small positive and negative values of $h$.

d) State all values of $a$ for which $f$ is not differentiable at $x=a$. For each, provide a reason for your conclusion.
e) True or false: if a function $p$ is differentiable at $x=b$, then $\lim _{x \rightarrow b} p(x)$ must exist. Why?

## V. Exercises

1. Consider the graph of the function $y=p(x)$ that is provided in the figure below. Assume that each portion of the graph of $p$ is a straight line, as pictured.


a) State all values of $a$ for which $\lim _{x \rightarrow a} p(x)$ does not exist.
b) State all values of $a$ for which $p$ is not continuous at $a$.
c) State all values of $a$ for which $p$ is not differentiable at $x=a$.
d) On the axes provided, sketch an accurate graph of $y=p^{\prime}(x)$.
2. For each of the following prompts, give an example of a function that satisfies the stated criteria. A formula or a graph, with reasoning, is sufficient for each. If no such example is possible, explain why.
a) A function $f$ that is continuous at $a=2$ but not differentiable at $a=2$.
b) A function $g$ that is differentiable at $a=3$ but does not have a limit at $a=3$.
c) A function $h$ that has a limit at $a=-2$, is defined at a $=-2$, but is not continuous at $a=-2$.
d) A function $p$ that satisfies all of the following:

- $p(-1)=3$ and $\lim _{x \rightarrow-1} p(x)=2$
- $p(0)=1$ and $p^{\prime}(0)=0$
- $\lim _{x \rightarrow 1} p(x)=p(1)$ and $p^{\prime}(1)$ does not exist

3. Let $h(x)$ be a function whose derivative $y=h^{\prime}(x)$ is given by the graph at the right.
a) Based on the graph of $y=h^{\prime}(x)$, what can you say about the behavior of the function $y=h(x)$ ?
b) At which values of $x$ is $y=h^{\prime}(x)$ not defined? What behavior does this lead you to expect to see in the graph of $y=h(x)$ ?

c) Is it possible for $y=h(x)$ to have points where h is not continuous? Explain your answer.
d) On the axes provided below, sketch at least two distinct graphs that are possible functions $y=h(x)$ that each have a derivative $y=h^{\prime}(x)$ that matches the provided graph at right. Explain why there are multiple possibilities for $y=h(x)$.

4. Consider the function $g(x)=\sqrt{|x|}$.
a) Use a graph to explain visually why $g$ is not differentiable at $x=0$.
b) Use the limit definition of the derivative to show that

$$
g^{\prime}(0)=\lim _{h \rightarrow 0} \frac{\sqrt{|h|}}{h}
$$


c) Investigate the value of $g^{\prime}(0)$ by estimating the limit in (b) using small positive and negative values of $h$. For instance, you might compute $\frac{\sqrt{|-0.01|}}{0.01}$. Be sure to use several different values of $h$ (both positive and negative), including ones closer to 0 than 0.01 . What do your results tell you about $g^{\prime}(0)$ ?

Use your graph in (a) to sketch an approximate graph of $y=g^{\prime}(x)$.
VI. Assessment - Khan Academy

1. Complete the first ten online practice exercises in the fourth unit (Derivative Introduction) of Khan Academy's AP Calculus AB course:
a. https://www.khanacademy.org/math/ap-calculus-ab/derivative-introduction-ab/differentiability-ab/e/differentiability-at-a-point-graphical
b. https://www.khanacademy.org/math/ap-calculus-ab/derivative-introduction-ab/differentiability-ab/e/differentiability-at-a-point-algebraic
II. Optional
2. None
