# 2.14 Let Me Take You Higher 

The Second Derivative



Given a differentiable function $y=f(x)$, you know that its derivative, $y=f^{\prime}(x)$, is a related function whose output at a value $x=a$ tells us the slope of the tangent line to $y=f(x)$ at the point $(a, f(a))$. That is, heights on the derivative graph tell us the values of slopes on the original function's graph. Therefore, the derivative tells us important information about the function $f$.


At any point where $f^{\prime}(x)$ is positive, it means that the slope of the tangent line to $f$ is positive, and therefore the function $f$ is increasing (or rising) at that point. Similarly, if $f^{\prime}$ (a) is negative, you know that the graph of $f$ is decreasing (or falling) at that point.

In this lesson, you will investigate whether the function $f$ is increasing or decreasing at a point or on an interval, but also how the function $f$ is increasing or decreasing. Comparing the two tangent lines shown above, you can see that at point $A$, the value of $f^{\prime}(x)$ is positive and relatively close to zero, which coincides with the graph rising slowly. By contrast, at point $B$, the derivative is negative and relatively large in absolute value, which is tied to the fact that $f$ is decreasing rapidly at B . It also makes sense to not only ask whether the value of the derivative function is positive or negative and whether the derivative is large or small, but also to ask "how is the derivative changing?"

Investigation 1: The position of a car driving along a straight road at time $t$ in minutes is given by the function $y=s(t)$ that is pictured below. The car's position function has units measured in thousands of feet. For instance, the point $(2,4)$ on the graph indicates that after 2 minutes, the car has traveled 4000 feet.

1a) In everyday language, describe the behavior of the car over the provided time interval.

In particular, discuss what is happening on each of the time intervals $[0,1],[1,2],[2,3],[3,4]$, and $[4,5]$, plus provide commentary overall on what the car is doing on the interval $[0,12]$.


b) On the graph at left, sketch a careful, accurate graph of $y=s^{\prime}(t)$.
c) What is the meaning of the function $y=s^{\prime}(t)$ in the context of the given problem? What can you say about the car's behavior when $s^{\prime}(t)$ is positive? when $s^{\prime}(t)$ is zero? when $s^{\prime}(t)$ is negative?
d) Rename the function you graphed in (b) to be called $y=v(t)$. Describe the behavior of $v$ in words, using phrases like " $v$ is increasing on the interval ..." and " $v$ is constant on the interval . ..."
e) Sketch a graph of the function $y=v^{\prime}(t)$ on the graph at right. Write at least one sentence to explain how the behavior of $v^{\prime}(t)$ is connected to the graph of $y=v(t)$.


## II. Increasing, decreasing, or neither

When you look at the graph of a function, there are features that strike us naturally, and common language can be used to name these features. In many different contexts, you have intuitively used the words increasing and decreasing to describe a function's graph.

Definition 2.1D. Given a function $f(x)$ defined on the interval $[a, b]$, you say that $f$ is increasing on $[a, b]$ provided that for all $(x, y)$ in the interval $[a, b]$, if $x<y$, then $f(x)<f(y)$. Similarly, you say that $f$ is decreasing on $[a, b]$, provided that for all $(x, y)$ in the interval $[a, b]$, if $x<y$, then $f(x)>f(y)$.

Simply put, an increasing function is one that is rising as you move from left to right along the graph, and a decreasing function is one that falls as the value of the input increases. For a function that has a derivative, you can use the sign of the derivative to determine whether or not the function is increasing or decreasing.

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A function that is decreasing on the intervals $-3<x<-2$ and $0<x<2$
and increasing on $-2<x<0$ and $2<x<3$.

## III. The Second Derivative

For any function, you should now be accustomed to investigating its behavior by thinking about its derivative. Given a function $f$, its derivative is a new function, one that is given by

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
$$

Because $f$ ' is itself a function, it is perfectly feasible for us to consider the derivative of the derivative, which is the new function $y=\left[f^{\prime}(x)\right]^{\prime}$. You call this resulting function the second derivative of $y=f(x)$, and denote the second derivative by $y=f$ " $(x)$. Due to the presence of multiple possible derivatives, you will sometimes call $f$ ' "the first derivative" of $f$, rather than simply "the derivative" of $f$. Formally, the second derivative is defined by the limit definition of the derivative of the first derivative:

$$
f^{\prime \prime}(x)=\lim _{h \rightarrow 0} \frac{f^{\prime}(x+h)-f^{\prime}(x)}{h}
$$

Notice that all of the established meaning of the derivative function still holds, so when you compute $y=f^{\prime \prime}(x)$, this new function measures slopes of tangent lines to the curve $y=f^{\prime}(x)$, as well as the instantaneous rate of change of $y=\mathrm{f}(x)$. In other words, just as the first derivative measures the rate at which the original function changes, the second derivative measures the rate at which the first derivative changes.

## Concavity

In addition to asking whether a function is increasing or decreasing, it is also natural to inquire how a function is increasing or decreasing. To begin, there are three basic behaviors that an increasing function can demonstrate on an interval, as pictured below: the function can increase more and more rapidly, increase at the same rate, or increase in a way that is slowing down.




Three functions that are all increasing, but doing so at an increasing rate, at a constant rate, and at a decreasing rate, respectively.




Consider the three graphs shown above. Clearly the middle graph demonstrates the behavior of a function decreasing at a constant rate.

For that function on the right, the slope of the tangent line is negative throughout the pictured interval, but as you move from left to right, the slopes get more and more negative. Hence the slope of the curve is decreasing, and you say that the function is decreasing at a decreasing rate.

The leftmost curve, on the other hand, is increasing at an increasing rate.
For that function, the slope of the tangent line is negative throughout the pictured interval, but as you move from left to right, the slopes get more and more negative. Hence the slope of the curve is decreasing, and you say that the function is decreasing at a decreasing rate.

This leads the notion of concavitywhich provides simpler language to describe some of these behaviors.


Concave Up


Concave Down

Informally, when a curve opens up on a given interval, like the upright parabola $y=x^{2}$ or the exponential growth function $y=e^{x}$, you say that the curve is concave up on that interval. Likewise, when a curve opens down, such as the parabola $y=-x^{2}$ or the opposite of
the exponential function $y=-e^{x}$, you say that the function is concave down. This behavior is linked to both the first and second derivatives of the function.

Investigation 2: The position of a car driving along a straight road at time $t$ in minutes is given by the function $y=s(t)$ that is pictured at right. The car's position function has units measured in thousands of feet. Remember that you worked with this function and sketched graphs of $y=v(t)=s^{\prime}(t)$ and $y=v^{\prime}(t)$ in Investigation 1.

2a) On what intervals is the position function $y=s(t)$ increasing? decreasing? Why?
b) On which intervals is the velocity function $v(t)=s^{\prime}(t)$
 increasing? decreasing? neither? Why?
c) Acceleration is defined to be the instantaneous rate of change of velocity, as the acceleration of an object measures the rate at which the velocity of the object is changing. Say that the car's acceleration function is named $a(t)$. How is $a(t)$ computed from $v(t)$ ? How is $a(t)$ computed from $s(t)$ ? Explain.
d) What can you say about $s$ " whenever $s^{\prime}$ is increasing? Why?
e) Using only the words increasing, decreasing, constant, concave up, concave down, and linear, complete the following sentences. For the position function $s$ with velocity $v$ and acceleration $a$,

- on an interval where $v$ is positive, $s$ is $\qquad$ .
- on an interval where $v$ is negative, $s$ is $\qquad$ .
- on an interval where $v$ is zero, $s$ is $\qquad$ .
- on an interval where $a$ is positive, $v$ is $\qquad$ .
- on an interval where $a$ is negative, $v$ is $\qquad$ .
- on an interval where $a$ is zero, $v$ is $\qquad$ .
- on an interval where $a$ is positive, $s$ is $\qquad$ .
- on an interval where $a$ is negative, $s$ is $\qquad$ .
- on an interval where $a$ is zero, $s$ is $\qquad$ .

The context of position, velocity, and acceleration is an excellent one in which to understand how a function, its first derivative, and its second derivative are related to one another.

Investigation 3: A potato is placed in an oven, and the potato's temperature $F$ (in degrees Fahrenheit) at various points in time is taken and recorded in the following table. Time $t$ is measured in minutes. In an earlier lesson, you computed approximations to $F^{\prime}(30)$ and $F$ '(60) using central differences. Those values and more are provided in the second table below, along with several others computed in the same way.

| $t$ | $F(t)$ |
| :---: | :---: |
| 0 | 70 |
| 15 | 180.5 |
| 30 | 251 |
| 45 | 296 |
| 60 | 324.5 |
| 75 | 342.8 |
| 90 | 354.5 |


| $t$ | $F^{\prime}(t)$ |
| :---: | :---: |
| 0 | N/A |
| 15 | 6.03 |
| 30 | 3.85 |
| 45 | 2,45 |
| 60 | 1.56 |
| 75 | 1.00 |
| 90 | N/A |

3a) What are the units on the values of $F^{\prime}(t)$ ?
b) Use a central difference to estimate the value of $F^{\prime \prime}(30)$.
c) What is the meaning of the value of $F^{\prime \prime}(30)$ that you have computed in (b) in terms of the potato's temperature? Write several careful sentences that discuss, with appropriate units, the values of $F(30), F^{\prime}(30)$, and $F$ " 30 ), and explain the overall behavior of the potato's temperature at this point in time.
d) Overall, is the potato's temperature increasing at an increasing rate, increasing at a constant rate, or increasing at a decreasing rate? Why?

Investigation 4: This activity builds on your experience and understanding of how to sketch the graph of $f^{\prime}$, given the graph of $f$.


4a) Given above, the respective graphs of two different functions $f^{\prime}$, sketch the corresponding graph of $f^{\prime}$ on the first axes below, and then sketch $f^{\prime \prime}$ on the second set of axes. In addition, for each, write several careful sentences in the spirit of those in Investigation 3 that connect the behaviors of $f$, $f^{\prime}$ and $f^{\prime \prime}$.

Throughout, view the scale of the grid for the graph of $f$ as being 1:1, and assume the horizontal scale of the grid for the graph of $f$ ' is identical to that for $f$. If you need to adjust the vertical scale on the axes for the graph of $f^{\prime}$ or $f^{\prime \prime}$ you should label that accordingly.

## IV. Exercises

1. Suppose that $y=f(x)$ is a differentiable function for which the following information is known: $f(2)=-3, f^{\prime}(2)=1.5, f^{\prime \prime}(2)=-0.25$.
a) Is $f$ increasing or decreasing at $x=2$ ? Is $f$ concave up or concave down at $x=2$ ?
b) Do you expect $f(2.1)$ to be greater than -3 , equal to -3 , or less than -3 ? Why?
c) Do you expect $f^{\prime}(2.1)$ to be greater than 1.5 , equal to 1.5 , or less than 1.5 ? Why?
d) Sketch a graph of $y=f(x)$ near $(2, f(2))$ and include a graph of the tangent.
2. For a certain function $y=g(x)$, the derivative is given by the function pictured below.
a) What is the approximate slope of the tangent line to $y=g(x)$ at the point $(2, g(2))$ ?
b) How many real number solutions can there be to the equation $g(x)=0$ ? Justify your conclusion fully and carefully by explaining what you know about how the
 graph of $g$ must behave based on the given graph of $g^{\prime}$
c) On the interval $-3<x<3$, how many times does the concavity of $g$ change? Explain how you know this.
d) Use the provided graph to estimate the value of $g$ "(2).
3. A bungee jumper's height $h$ (in feet) at time $t$ (in seconds) is given in part by the data in the following table:

| $t$ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $h(t)$ | 200 | 184.2 | 159.9 | 131.9 | 104.7 | 81.8 | 65.5 | 56.8 | 55.5 | 60.4 | 69.8 |


| $t$ | 5.5 | 6.0 | 6.5 | 7.0 | 7.5 | 8.0 | 8.5 | 9.0 | 9.5 | 10.0 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $h(t)$ | 81.6 | 93.7 | 104.4 | 112.6 | 117.7 | 119.4 | 118.2 | 114.8 | 110.0 | 104.7 |

a) Use the given data to estimate $h^{\prime}(4.5), h^{\prime}(5)$, and $h^{\prime}(5.5)$. At which of these times is the bungee jumper rising most rapidly?
b) Use the given data and your work in (a) to estimate $h$ "(5).
c) What physical property of the bungee jumper does the value of $h$ " $(5)$ measure? What are its units?
d) Based on the data, on what approximate time intervals is the function $y=h(t)$ concave down? What is happening to the velocity of the bungee jumper on these time intervals?
4. For each prompt that follows, sketch a possible graph of a function on the interval $-3<x<3$ that satisfies the stated properties.
a) $y=f(x)$ such that $f$ is increasing on $-3<x<3, f$ is concave up on $-3<x<0$, and $f$ is concave down on $0<x<3$.
b) $y=g(x)$ such that $g$ is increasing on $-3<x<3, \mathrm{~g}$ is concave down on $-3<x<0$, and $g$ is concave up on $0<x<3$.
c) $y=h(x)$ such that $h$ is decreasing on $-3<x<3$, his concave up on $-3<x<1$, neither concave up nor concave down on $-1<x<1$, and h is concave down on $1<$ $x<3$.
d) $y=p(x)$ such that $p$ is decreasing and concave down on $-3<x<0$ and $p$ is increasing and concave down on $0<x<3$.

## V. Assessment - Khan Academy

1. None
