### 2.13 Get Real

Interpreting, Estimating and Using the Derivative

An interesting and powerful feature of mathematics is that it can often be thought of both in abstract terms and
 in applied ones. For instance, calculus can be developed almost entirely as an abstract collection of ideas that focus on properties of arbitrary functions. At the same time, calculus can also be very directly connected to your experience of physical reality by considering functions that represent meaningful processes.

In this lesson, you will investigate several different functions, each with specific physical meaning, and think about how the units on the independent variable, dependent variable, and the derivative function add to your understanding. To start, consider the familiar problem of a position function of a moving object.

Investigation 1: One of the longest stretches of straight (and flat) road in North America can be found on the Great Plains in the state of North Dakota on state highway 46, which lies just south of the interstate highway I-94 and runs through the town of Gackle (picture above, courtesy of Kadim Alasady from his bike tour web site).


A car leaves town (at time $t=0$ ) and heads east on highway 46; its position in miles from Gackle at time $t$ in minutes is given by the graph of the function at left. Three important points are labeled on the graph; where the curve looks linear, assume that it is indeed a straight line.

1a) In everyday language, describe the behavior of the car over the provided time interval. In particular, discuss what is happening on the time intervals $[57,68]$ and $[68$, 104].
b) Find the slope of the line between the points $(57,63.8)$ and $(104,106.8)$. What are the units on this slope? What does the slope represent?
c) Find the average rate of change of the car's position on the interval [68, 104]. Include units on your answer.
d) Estimate the instantaneous rate of change of the car's position at the moment $t=80$. Write a sentence to explain your reasoning and the meaning of this value.

## II. Units of the derivative function

As you now know, the derivative of the function $f$ at a fixed value $x$ is given by

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
$$

and this value has several different interpretations. If you set $x=a$, one meaning of $f^{\prime}(a)$ is the slope of the tangent line at the point ( $\mathrm{a}, \mathrm{f}(\mathrm{a})$ ).

In alternate notation, you also sometimes equivalently write $\frac{d f}{d x}$ or $\frac{d y}{d x}$ instead of $f^{\prime}(x)$, and these notations helps us to further see the units (and thus the meaning) of the derivative as it is viewed as the instantaneous rate of change of $f$ with respect to $x$. Note that the units on the slope of the secant line, $\frac{f(x+h)-f(x)}{h}$, are "units of f per unit of x ." Thus, when you take the limit to get $\mathrm{f}^{\prime}(\mathrm{x})$, you get these same units on the derivative $f^{\prime}(x)$ : units of $f$ per unit of $x$. Regardless of the function $f$ under consideration (and regardless of the variables being used), it is helpful to remember that the units on the derivative function are "units of output per unit of input," in terms of the input and output of the original function.

For example, say that you have a function $y=P(t)$, where $P$ measures the population of a city (in thousands) at the start of year $t$ (where $t=0$ corresponds to 2010 AD ), and you are told that $P^{\prime}(2)=21.37$. What is the meaning of this value? Well, since $P$ is measured in thousands and $t$ is measured in years, you can say that the instantaneous rate of change of the city's population with respect to time at the start of 2012 is 21.37 thousand people per year. You therefore expect that in the coming year, about 21,370 people will be added to
the city's population.

## Toward more accurate derivative estimates

It is also helpful to recall, as you first experienced in an earlier lesson, that when you want to estimate the value of $f^{\prime}(x)$ at a given $x$, you can use the difference quotient $\frac{f(x+h)-f(x)}{h}$ with a relatively small value of $h$. In doing so, you should use both positive and negative values of $h$ to make sure you account for the behavior of the function on both sides of the point of interest. To that end, consider the following brief example to demonstrate the notion of a central difference and its role in estimating derivatives.

Example 1. Suppose that $y=f(x)$ is a function for which three values are known: $f(1)=2.5, f(2)=3.25$, and $f(3)=3.625$. Estimate $f^{\prime}(2)$.

Solution. You know that $f^{\prime}(2)=\lim _{h \rightarrow 0} \frac{f(2+h)-f(2)}{h}$. But since you don't have a graph for $y=f(x)$ nor a formula for the function, you can neither sketch a tangent line nor evaluate the limit exactly. You can't even use smaller and smaller values of $h$ to estimate the limit. Instead, you have just two choices: using $h=-1$ or $h=1$, depending on which point you pair with $(2,3.25)$.

So, one estimate is

$$
f^{\prime}(2) \approx \lim _{h \rightarrow 0} \frac{f(1)-f(2)}{1-2}=\frac{2.5-3.25}{-1}=0.75
$$

The other is

$$
f^{\prime}(2) \approx \lim _{h \rightarrow 0} \frac{f(3)-f(2)}{3-2}=\frac{3.625-3.25}{1}=0.375
$$

Since the first approximation looks only backward from the point $(2,3.25)$ and the second approximation looks only forward from ( $2,3.25$ ), it makes sense to average these two values in order to account for behavior on both sides of the point of interest. Doing so, you find that

$$
f^{\prime}(2) \approx \frac{0.75+0.375}{2}=0.5625
$$

The intuitive approach to average the two estimates found in Example 1 is in fact the best possible estimate to $f^{\prime}(2)$ when you have just two function values for $f$ on opposite sides of the point of interest.

To see why, think about the diagram below, which shows a possible function $y=f(x)$ that satisfies the data given in Example 1.


On the left are the two secant lines with slopes that come from computing the backward difference $\frac{f(1)-f(2)}{1-2}=0.75$ and from the forward difference $\frac{f(3)-f(2)}{3-2}=0.375$. Notice how the first such line's slope over-estimates the slope of the tangent line at $(2, f(2))$, while the second line's slope underestimates $f^{\prime}(2)$.

On the right, however, you see the secant line whose slope is given by the central difference $\frac{f(3)-f(1)}{3-1}=\frac{0.75+0.375}{2}=0.5625$. Notice that this central difference has the exact same value as the average of the forward difference and backward difference (and it is straightforward to explain why this always holds), and moreover that the central difference yields a very good approximation to the derivative's value, in part because the secant line that uses both a point before and after the point of tangency yields a line that is closer to being parallel to the tangent line.

In general, the central difference approximation to the value of the first derivative is given by

$$
f^{\prime}(a) \approx \frac{f(a+h)-f(a-h)}{2 h}
$$

and this quantity measures the slope of the secant line to $y=f(x)$ through the points ( $a-h, f(a-h)$ ) and $(a+h, f(a+h)$ ). Anytime you have symmetric data surrounding a point at which you desire to estimate the derivative, the central difference is an ideal choice for so doing.

The following activities will further explore the meaning of the derivative in several different contexts while also viewing the derivative from graphical, numerical, and algebraic perspectives.

Investigation 2: A potato is placed in an oven, and the potato's temperature F (in degrees Fahrenheit) at various points in time is taken and recorded in the following table. Time $t$ is measured in minutes.

| $t$ | $F(t)$ |
| :---: | :---: |
| 0 | 70 |
| 15 | 180.5 |
| 30 | 251 |
| 45 | 296 |
| 60 | 324.5 |
| 75 | 342.8 |
| 90 | 354.5 |

a) Use a central difference to estimate the instantaneous rate of change of the temperature of the potato at $t=30$. Include units on your answer.
b) Use a central difference to estimate the instantaneous rate of change of the temperature of the potato at $t=60$. Include units on your answer.
c) Without doing any calculation, which do you expect to be greater: $F^{\prime}(75)$ or $F^{\prime}(90)$ ? Why?
d) Suppose it is given that $F(64)=330.28$ and $F^{\prime}(64)=1.341$. What are the units on these two quantities? What do you expect the temperature of the potato to be when $t=65$ ? when $t=66$ ? Why?
e) Write a couple of careful sentences that describe the behavior of the temperature of the potato on the time interval [ 0,90 ], as well as the behavior of the instantaneous rate of change of the temperature of the potato on the same time interval.

Investigation 3: A company manufactures rope, and the total cost of producing $r$ feet of rope is $C(r)$ dollars.
a) What does it mean to say that $C(2000)=800$ ?
b) What are the units of $C^{\prime}(r)$ ?
c) Suppose that $C(2000)=800$ and $C^{\prime}(2000)=0.35$. Estimate $C(2100)$, and justify your estimate by writing at least one sentence that explains your thinking.
d) Which of the following statements do you think is true, and why?

$$
\text { i. } C^{\prime}(2000)<C^{\prime}(3000)
$$

ii. $C^{\prime}(2000)=C^{\prime}(3000)$
iii. $C^{\prime}(2000)>C^{\prime}(3000)$
e) Suppose someone claims that $C^{\prime}(5000)=0.1$. What would the practical meaning of this derivative value tell you about the approximate cost of the next foot of rope? Is this possible? Why or why not?

Investigation 4: Researchers at a major car company have found a function that relates gasoline consumption to speed for a particular model of car. In particular, they have determined that the consumption $C$, in liters per kilometer, at a given speed $s$, is given by a function $C=f(s)$, where $s$ is the car's speed in kilometers per hour.
a) Data provided by the car company tells us that $f(80)=0.015, f(90)=0.02$, and $f(100)=0.027$. Use this information to estimate the instantaneous rate of change of fuel consumption with respect to speed at $s=90$. Be as accurate as possible, use proper notation, and include units on your answer.
b) By writing a complete sentence, interpret the meaning (in the context of fuel consumption) of $f(80)=0.015$.
c) Write at least one complete sentence that interprets the meaning of the value of $f^{\prime}(90)$ that you estimated in (a).

In the last lesson, you learned how use to the graph of a given function $f$ to plot the graph of its derivative, $f^{\prime}$. It is important to remember that when you do so, not only does the scale on the vertical axis often have to change to accurately represent $f^{\prime}$ but the units on that axis also differ. For example, suppose that $P(t)=400330 \mathrm{e}^{-0.03 t}$ tells us the temperature in degrees Fahrenheit of a potato in an oven at time $t$ in minutes. In the figure below, examine the graph of $P$ on the left and the graph of $P$ ' on the right.


Notice how the vertical scales differ in both size and units, as the units of $P$ are ${ }^{\circ} \mathrm{F}$, while those of $P$, are ${ }^{\circ} \mathrm{F} / \mathrm{min}$. In all cases where you work with functions that have an applied context, it is helpful and instructive to think carefully about units involved and how they further inform the meaning of your computations.

## III. Exercises

1. A cup of coffee has its temperature $F$ (in degrees Fahrenheit) at time $t$ given by the function $F(t)=75+110 e^{-0.05 t}$, where time is measured in minutes.
a) Use a central difference with $h=0.01$ to estimate the value of $F^{\prime}(10)$.
b) What are the units on the value of $F^{\prime}(10)$ that you computed in (a)? What is the practical meaning of the value of $F^{\prime}(10)$ ?
c) Which do you expect to be greater: $F^{\prime}(10)$ or $F^{\prime}(20)$ ? Why?
d) Write a sentence that describes the behavior of the function $y=F^{\prime}(t)$ on the time interval $0 \leq t \leq 30$. How do you think its graph will look? Why?
2. The temperature change $T$ (in Fahrenheit degrees), in a patient, that is generated by a dose $q$ (in milliliters), of a drug, is given by the function $T=f(q)$.
a) What does it mean to say $f(50)=0.75$ ? Write a complete sentence to explain, using correct units.
b) A person's sensitivity, $s$, to the drug is defined by the function $s(q)=f^{\prime}(q)$. What are the units of sensitivity?
c) Suppose that $f^{\prime}(50)=0.02$. Write a complete sentence to explain the meaning of this value. Include in your response the information given in (a).
3. The velocity of a ball that has been tossed vertically in the air is given by $v(t)=16-32 t$, where $v$ is measured in feet per second, and $t$ is measured in seconds. The ball is in the air from $t=0$ until $t=2$.
a) When is the ball's velocity greatest?
b) Determine the value of $v^{\prime}(1)$. Justify your thinking.
c) What are the units on the value of $v^{\prime}(1)$ ? What does this value and the corresponding units tell you about the behavior of the ball at time $t=1$ ?
d) What is the physical meaning of the function $v^{\prime}(\mathrm{t})$ ?
4. The value, V , of a particular automobile (in dollars) depends on the number of miles, m , the car has been driven, according to the function $V=h(m)$.
a) Suppose that $h(40000)=15500$ and $h(55000)=13200$. What is the average rate of change of $h$ on the interval [40000, 55000], and what are the units on this value?
b) In addition to the information given in (a), say that $h(70000)=11100$.

Determine the best possible estimate of $h^{\prime}(55000)$ and write one sentence to explain the meaning of your result, including units on your answer.
c) Which value do you expect to be greater: $h^{\prime}(30000)$ or $h^{\prime}(80000)$ ? Why?
d) Write a sentence to describe the long-term behavior of the function $V=h(m)$, plus another sentence to describe the long-term behavior of $h^{\prime}(m)$. Provide your discussion in practical terms regarding the value of the car and the rate at which that value is changing.

## IV.Assessment - Khan Academy

1. None
