### 2.11 That's So Derivative

Introduction to Differential Calculus



Just as one defines instantaneous velocity in terms of average velocity, we now define the instantaneous rate of change of a function at a point in terms of the average rate of change of the function $f$ over related intervals. In addition, this quantity is called "the derivative of $f$ at $a, "$ with this value being represented by the shorthand notation $f^{\prime}(a)$.

Definition 2.1A. Let $f$ be a function and $x=a$, a value in the function's domain. We define the derivative of $f$ with respect to $x$ evaluated at $x=a$, denoted $f^{\prime}(a)$, by the formula,

$$
f^{\prime}(a)=\lim _{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}
$$

provided this limit exists.

Aloud, you may read the symbol $f^{\prime}(a)$ as either " $f$-prime at $a$ " or "the derivative of $f$ evaluated at $x=a$." Here are some important notes to keep in mind:

- The derivative of $f$ at the value $x=a$ is defined as the limit of the average rate of change of $f$ on the interval $[a, a+h]$ as $h \rightarrow 0$. It is possible for this limit not to exist, so not every function has a derivative at every point.
- One says that a function that has a derivative at $x=a$ is differentiable at $x=a$.
- The derivative is a generalization of the instantaneous velocity of a position function: when $y=s(t)$ is a position function of a moving body, $s$, a tells us the instantaneous velocity of the body at time $t=a$.
- Because the units on $\frac{f(a+h)-f(a)}{h}$ are "units of $f$ per unit of $x$," the derivative has these
very same units. For instance, if $s$ measures position in feet and $t$ measures time in seconds, the units on $s^{\prime}(a)$ are feet per second.
- Because the quantity $\frac{f(a+h)-f(a)}{h}$ represents the slope of the line through ( $a, f(a)$ ) and ( $a+h, f(a+h)$ ), when you compute the derivative you are taking the limit of a collection of slopes of lines, and thus the derivative itself represents the slope of a particularly important line.

As you move from an average rate of change to an instantaneous one, think of one point as "sliding towards" another.

Investigation 1: Investigate the GeoGebra app at https://ggbm.at/Ndej8dKI
1a) In your own words, describe what happens to the point $(a+h, f(a+h)$ as $h \rightarrow 0$.


In the figure above, notice the sequence of figures with several different lines through the points $(a, f(a))$ and $(a+h, f(a+h))$, that are generated by different values of $h$. These lines (shown in the first three figures in magenta), are often called secantlines to the curve $y=f(x)$. A secant line to a curve is simply a line that passes through two points that lie on the curve. For each such line, the slope of the secant line $\frac{f(a+h)-f(a)}{h}$, where the value of $h$ depends on the location of the point you choose.

Return to the GeoGebra app and Click on "Show secant lines".
b) When $h \rightarrow 0$, what happens to the slope of the secant line?
c) How would this compare to a tangent line at $x=a$ ?

It is most important to note that $f^{\prime}(a)$, the instantaneous rate of change of $f$ with respect to $x$ at $x=a$, also measures the slope of the tangent line to the curve $y=f(x)$ at $(a, f(a))$. The following example demonstrates several key ideas involving the derivative of a function.

Example 1. For the function given by $f(x)=x-x^{2}$, use the limit definition of the derivative to compute $f^{\prime}$ (2). In addition, discuss the meaning of this value and draw a labeled graph that supports your explanation.

Solution. From the limit definition, you know that
$f^{\prime}(2)=\lim _{h \rightarrow 0} \frac{f(2+h)-f(2)}{h}$
Now you use the rule for $f$, and observe that $\mathrm{f}(2)=2-2^{2}=-2$ and $\mathrm{f}(2+h)=$ $(2+h)-(2+h)^{2}$. Substituting these values into the limit definition, you have that

$$
f^{\prime}(2)=\lim _{h \rightarrow 0} \frac{(2+h)-(2+h)^{2}-(-2)}{h}
$$

Observe that with $h$ in the denominator and our desire to let $h \rightarrow 0$, you have to wait to take the limit (that is, you wait to actually let h approach 0 ). Thus, you do additional algebra. Expanding and distributing the numerator results in

$$
f^{\prime}(2)=\lim _{h \rightarrow 0} \frac{2+h-4-4 h-h^{2}+2}{h}
$$

Combining like terms, you get

$$
f^{\prime}(2)=\lim _{h \rightarrow 0} \frac{-3 h-h^{2}}{h}
$$

Next, observe that there is a common factor of $h$ in both the numerator and denominator, which allows us to simplify and find that

$$
f^{\prime}(2)=\lim _{h \rightarrow 0}-3-h
$$

Finally, you are able to take the limit as $h \rightarrow 0$, and thus conclude that $f^{\prime}(2)=-3$.
Now, you know that $f^{\prime}(2)$ represents the slope of the tangent line to the curve $y=x-x^{2}$ at the point $(2,-2)$; $f^{\prime}(2)$ is also the instantaneous rate of change of $f$ at the point $(2,-2)$.

Graphing both the function and the line through $(2,-2)$ with slope $m=f^{\prime}(2)=3$, you indeed see that by calculating the derivative, you have found the slope of the tangent line at this point, as shown in the figure below.


The following investigation will help you explore a variety of key ideas related to derivatives.

Investigation 2: Consider the function $f$ whose formula is $f(x)=3-2 x$.
2a) What familiar type of function is $f$ ? What can you say about the slope of $f$ at every value of $x$ ?
b) Compute the average rate of change of $f$ on the intervals $[1,4],[3,7]$, and $[5,5+h]$; simplify each result as much as possible. What do you notice about these quantities?
c) Use the limit definition of the derivative to compute the exact instantaneous rate of change of $f$ with respect to $x$ at the value $a=1$. That is, compute $f^{\prime}(1)$ using the limit definition. Show your work. Is your result surprising?

Without doing any additional computations, what are the values of $f^{\prime}(2), f^{\prime}(\pi)$, and $f^{\prime}(\sqrt{2})$ ? Why?

Investigation 3: A water balloon is tossed vertically in the air from a window. The balloon's height in feet at time $t$ in seconds after being launched is given by $s(t)=16 t^{2}+16 t+32$. Use this function to respond to each of the following questions.

3a) Sketch an accurate, labeled graph of $s$ on the axes provided below. You should be able to do this without using computing technology.

b) Compute the average rate of change of $s$ on the time interval [1, 2]. Include units on your answer and write one sentence to explain the meaning of the value you found.
c) Use the limit definition to compute the instantaneous rate of change of $s$ with respect to time, $t$, at the instant $a=1$. Show your work using proper notation, include units on your answer, and write one sentence to explain the meaning of the value you found.
d) On your graph in (a), sketch two lines: one whose slope represents the average rate of change of $s$ on 1,2 , the other whose slope represents the instantaneous rate of change of $s$ at the instant $a=1$. Label each line clearly.
e) For what values of $a$ do you expect $s^{\prime}(a)$ to be positive? Why? Answer the same questions when "positive" is replaced by "negative" and "zero."

Investigation 4: A rapidly growing city in Arizona has its population $P$ at time $t$, where $t$ is the number of decades after the year 2010, modeled by the formula $P(t)=25000 e^{t / 5}$. Use
this function to respond to the following questions.
4a) Sketch an accurate graph of $P$ for $t=0$ to $t=5$ on the axes provided below. Label the scale on the axes carefully

b) Compute the average rate of change of $P$ between 2030 and 2050. Include units on your answer and write one sentence to explain the meaning (in everyday language) of the value you found.
c) Use the limit definition to write an expression for the instantaneous rate of change of $P$ with respect to time, $t$, at the instant $a=2$. Explain why this limit is difficult to evaluate exactly.
d) Estimate the limit in (c) for the instantaneous rate of change of $P$ at the instant $a=2$ by using several small $h$ values. Once you have determined an accurate estimate of $P^{\prime}(2)$, include units on your answer, and write one sentence (using everyday language) to explain the meaning of the value you found.
e) On your graph above, sketch two lines: one whose slope represents the average rate of change of $P$ on $[2,4]$, the other whose slope represents the instantaneous rate of change of $P$ at the instant $a=2$.
f) In a carefully-worded sentence, describe the behavior of $P^{\prime}(a)$ as $a$ increases in value. What does this reflect about the behavior of the given function $P$ ?

## II. Exercises

1. Consider the graph of $y=f(x)$ provided at right.
a) On the graph of $y=f(x)$, sketch and label the following quantities:

- the secant line to $y=f(x)$ on the interval [-3,1] and the secant line to $y=f(x)$ on the interval [0,2].

- the tangent line to $y=f(x)$ at $x=-3$ and the tangent line to $y=f(x)$ at $x=0$.
b) What is the approximate value of the average rate of change of $f$ on $[-3,-1]$ ? On $[0,2]$ ? How are these values related to your work in (a)?
c) What is the approximate value of the instantaneous rate of change of $f$ at $x=-3$ ? At $x$ $=0$ ? How are these values related to your work in (a)?

2. For each of the following prompts, sketch a graph on the provided axes of a function that has the stated properties.
a) $y=f(x)$ such that

- the average rate of change of $f$ on $[-3,0]$ is -2 and the average rate of change of $f$ on $[1,3]$ is 0.5 , and

- the instantaneous rate of change of $f$ at $x$ $=1$ is 1 and the instantaneous rate of change of f at $x=2$ is 1 .
b) $y=g(x)$ such that
- $\frac{g(3)-g(-2)}{5}=0$ and $\frac{g(1)-g(-1)}{2}=-1$, and
- $g^{\prime}(2)=1$ and $g^{\prime}(2)=1$


3. Suppose that the population, $P$, of China (in billions) can be approximated by the function $P(t)=1.15(1.014)^{t}$ where $t$ is the number of years since the start of 1993.
a) According to the model, what was the total change in the population of China between January 1, 1993 and January 1, 2000? What will be the average rate of change of the population over this time period? Is this average rate of change greater or lesser than the instantaneous rate of change of the population on January 1, 2000? Explain and justify, being sure to include proper units on all your answers.
b) According to the model, what is the average rate of change of the population of China in the ten-year period starting on January 1, 2012?
c) Write an expression involving limits that, if evaluated, would give the exact instantaneous rate of change of the population on today's date. Then estimate the value of this limit (discuss how you chose to do so) and explain the meaning (including units) of the value you have found.
d) Find an equation for the tangent line to the function $y=P(t)$ at the point where the $t$ value is given by today's date.
4. The goal of this problem is to compute the value of the derivative at a point for several different functions, in three different ways, and then compare the results.

For each of the following functions, use the limit definition of the derivative to compute the value of $f^{\prime}(a)$ using three different approaches: strive to use the algebraic approach first (to compute the limit exactly), then test your result using numerical evidence (with small values of $h$ ), and finally plot the graph of $y=f(x)$ near ( $a, f(a)$ ) along with the appropriate tangent line to estimate the value of $f$, a visually. Compare your findings among all three approaches; if you are unable to complete the algebraic approach, still work numerically and graphically.
a) $f(x)=x^{2}-3 x, a=2$
b) $f(x)=1, a=1$
c) $f(x)=\sqrt{x} a=1$
d) $f(x)=2-|x-1|, a=1$
e) $f(x)=\sin (x), a=\frac{\pi}{2}$
III. Assessment - Khan Academy

1. Complete the first 10 practice exercises from the Derivative Introduction unit in Khan Academy's AP Calculus AB course: https://www.khanacademy.org/math/ap-calculus-ab/derivative-introduction-ab?t=practice
2. Optional: Practice \#11 (Challenge)
