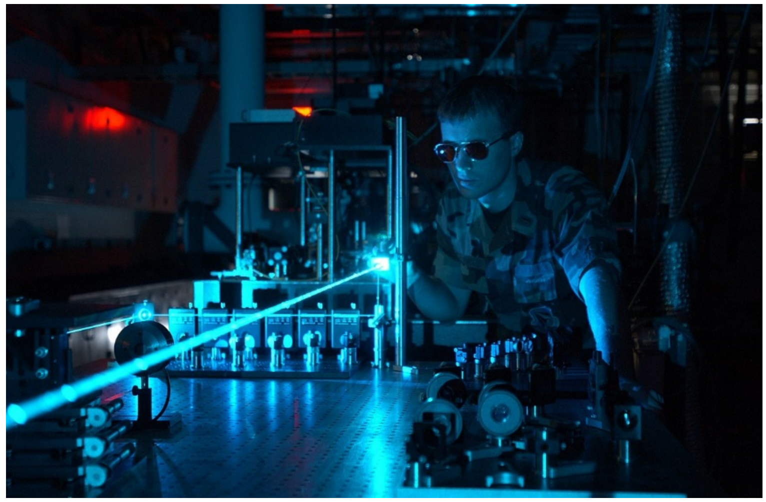
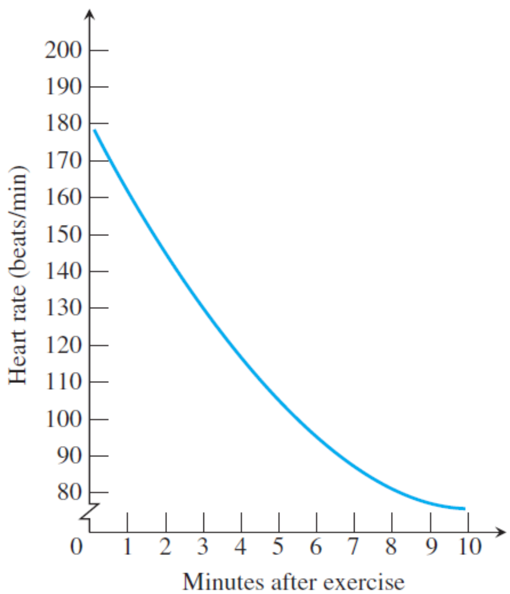
1.5 Continuity



Continuous Functions

When you plot function values generated in the laboratory or collected in the field, you often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the times we did not measure.



In doing so, you are assuming that you are working with a continuous function, a function whose outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between.

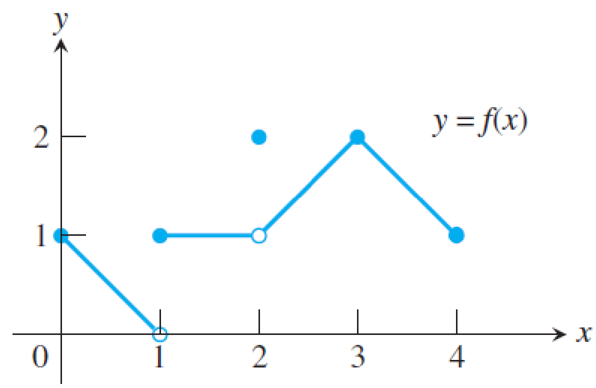
Any function $y = f(x)$ whose graph can be sketched in one continuous motion without lifting the pencil is an example of a ***continuous function***.

Continuous functions are the functions used to find a planet's closest point of approach to the sun or the peak concentration of antibodies in blood plasma. They are also the functions used to describe how a body moves through space or how the speed of a chemical reaction changes with time.

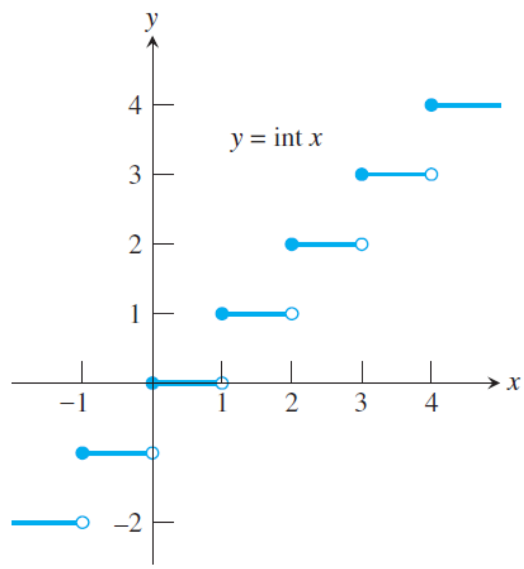
In fact, so many physical processes proceed continuously that throughout the eighteenth and nineteenth centuries it rarely occurred to anyone to look for any other kind of behavior. It came as a surprise when the physicists of the 1920s discovered that light comes in particles and that heated atoms emit light at discrete frequencies

As a result of these and other discoveries, and because of the heavy use of discontinuous functions in computer science, statistics, and mathematical modeling, the issue of continuity has become one of practical as well as theoretical importance.

Investigation 1: Find the points at which the function f (pictured at right) is continuous, and the points at which f is discontinuous.



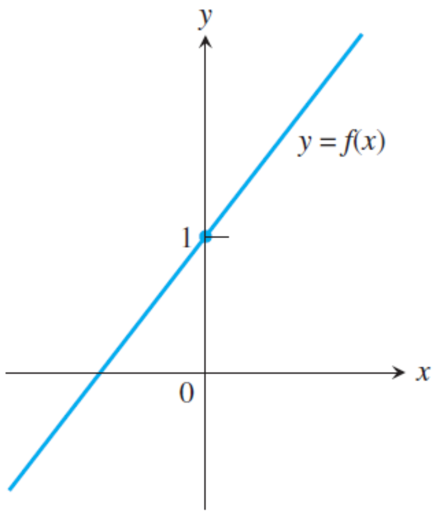
If a function f is not continuous at a point c, we say that f is ***discontinuous*** at c and c is a point of discontinuity of f. Note that c need not be in the domain of f.



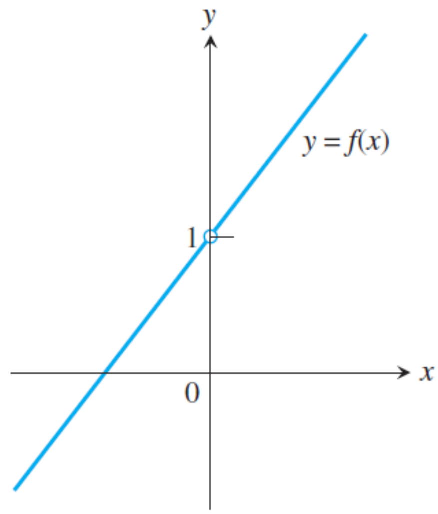
1b) Find the points of continuity and the points of discontinuity of the greatest integer function (pictured above.)

II. Types of Discontinuity

Pictured below is a catalog of discontinuity types.



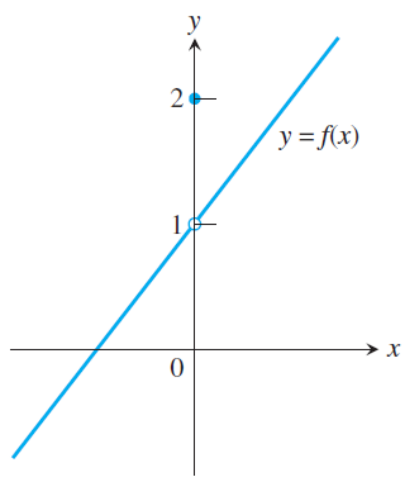
(a)



(b)

The function in (a) is continuous at $x = 0$.

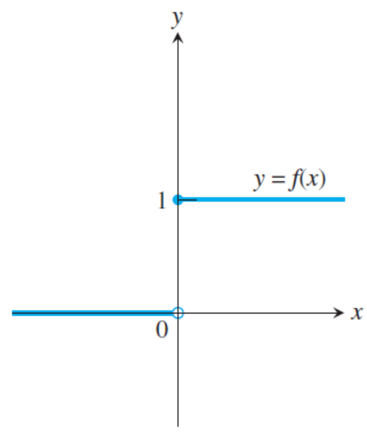
The function in (b) would be if it had $f(0) = 1$.



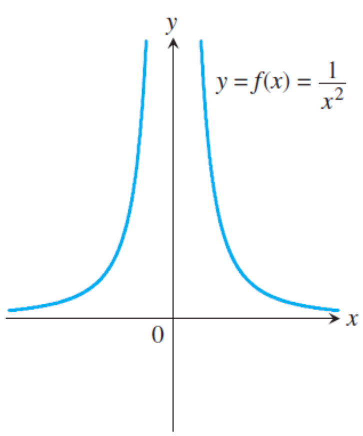
(c)

The function in (c) would be continuous if $f(0)$ were 1 instead of 2. The discontinuities in (b) and (c) are removable.

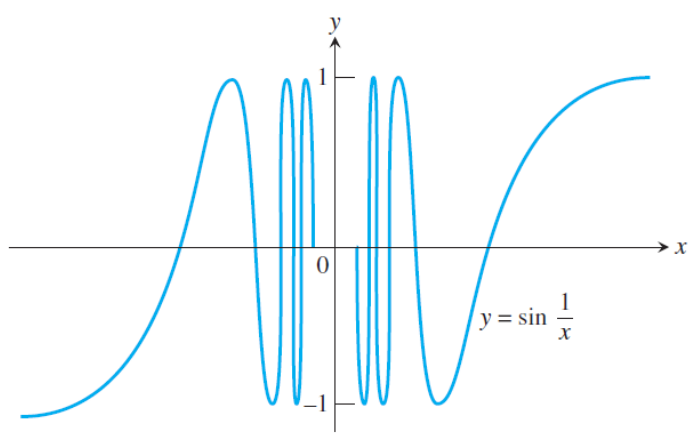
Each function has a limit as $x \to 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.



(d)



(e)



(f)

The discontinuities in (d)–(f) are more serious: $\lim_{x \to 0} f(x)$ does not exist and there is no way to improve the situation by changing *f* at 0.

- The step function in (d) has a ***jump discontinuity***: the one-sided limits exist but have different values.
- The function $f(x) = \frac{1}{x^2}$ in (e) has an ***infinite discontinuity***.
- The function in (f) has an ***oscillating discontinuity***: it oscillates and has no limit as $x \to 0$.

Investigation 2: Removing a Discontinuity

Let $f(x) = \frac{x^3-7x-6}{x^2-9}$.

a) Factor the denominator. What is the domain of f?

b) Investigate the graph of f around $x = 3$ to see that f has a removable discontinuity at $x = 3$.

c) How should f be defined at $x = 3$ to remove the discontinuity? Use zoom-in and tables as necessary.

d) Show that $(x - 3)$ is a factor of the numerator of f, and remove all common factors.

e) Now compute the limit as $x \to 3$ of the reduced form for f.

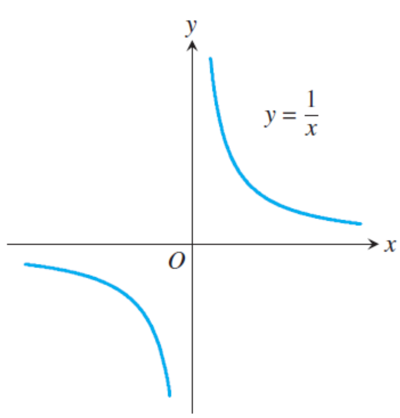
f) Show that the extended function

$$g(x) = \begin{cases} \dfrac{x^3 - 7x - 6}{x^2 - 9}, & x \neq 3 \\ \dfrac{10}{3}, & x = 3 \end{cases}$$

is continuous at $x = 3$. The function g is the continuous extension of the original function f to include $x = 3$.

III. Continuous Functions

A function is ***continuous on an interval*** if and only if it is continuous at every point of the interval. A ***continuous function*** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval. For example, $y = \frac{1}{x}$ is not continuous on [-1, 1].



The reciprocal function $y = \frac{1}{x}$ (at left) is a continuous function because it is continuous at every point of its domain. However, it has a point of discontinuity at $x = 0$ because it is not defined there.

Polynomial functions f are continuous at every real number c because $\lim_{x \to c} f(x) = f(c)$.

Rational functions are continuous at every point of their domains. They have points of discontinuity at the zeros of their denominators. The absolute value function $y = |x|$ is continuous at every real number. The exponential functions, logarithmic functions, trigonometric functions, and radical functions like $y = \sqrt[n]{x}$ (n a positive integer greater than 1) are continuous at every point of their domains. All of these functions are continuous functions!

IV. Practice

1. Find the points of discontinuity and identify the type of discontinuity.

a) $y = \frac{|x|}{x}$

b) $y = \frac{1}{(x+2)^2}$

c) $y = \cot x$

d) $y = e^{1/x}$

2. Find each point of discontinuity. Which of the discontinuities are removable? Explain how you know.

a) $g(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & x > 2 \end{cases}$

b) $h(x) = \begin{cases} 3 - x, & x < 2 \\ 3 - x, & x = 2 \\ \frac{x}{2}, & x > 2 \end{cases}$

c) $g(x) = \begin{cases} 1 - x^2, & x \neq -1 \\ 2, & x = -1 \end{cases}$

V. Practice – Khan Academy

Complete the next three online practice exercises in the first unit (Continuity) of Khan Academy's AP Calculus AB course:

- https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-continuity/ab-discontinuities/e/analyzing-discontinuities-graphical
- https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-continuity/ab-discontinuities/e/limits-of-piecewise-functions
- https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-continuity/ab-discontinuities/e/continuity