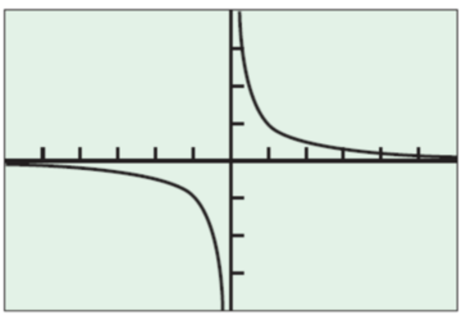
1.4 Limits Involving Infinity

Asymptotes and End Behavior

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, when one says "the limit of f as x approaches infinity" it is meant the limit of f as x moves increasingly far to the right on the number line. When we say "the limit of f as x approaches negative infinity ($-\infty$)" we mean the limit of f as x moves increasingly far to the left. (The limit in each case may or may not exist.)



Looking at $f(x) = \frac{1}{x}$, (pictured at left), observe that

(a) as $x \to \infty, \frac{1}{x} \to 0$ and you would write

$$\lim_{x\to\infty} \left(\frac{1}{x}\right) = 0$$

(b) as $x \to -\infty, \frac{1}{x} \to 0$ and you would write

$$\lim_{x\to\infty} \left(\frac{1}{x}\right) = 0.$$

Therefore, the line $y = 0$ is a ***horizontal asymptote*** of the graph of f.

Definition: Horizontal Asymptote

The line $y = b$ is a horizontal asymptote of the graph of a function $f(x)$ if either $\lim_{x\to\infty} f(x) = b,$ or $\lim_{x\to-\infty} f(x) = b.$

The graph of $f(x)$ has the single horizontal asymptote $y = 2$ because

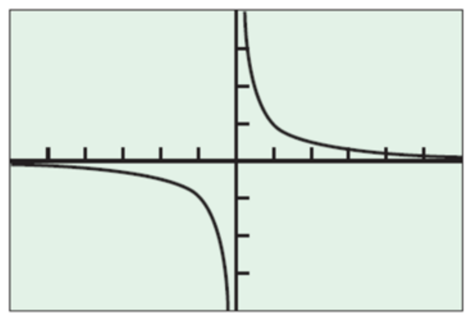
$$\lim_{x\to\infty}\left(2+\frac{1}{x}\right)=2 \quad and \quad \lim_{x\to-\infty}\left(2+\frac{1}{x}\right)=2$$

Investigation 1: Use graphs and tables to find $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$, and identify all the horizontal asymptotes of $f(x)=\frac{x}{\sqrt{x^2+1}}$.

II. Infinite Limits as $x \to a$

If the values of a function $f(x)$ outgrow all positive bounds as x approaches a finite number a, one says that $\lim_{x\to a} f(x)=\infty$. If the values of f become large and negative, exceeding all negative bounds as $x\to a$, it is said that $\lim_{x\to a} f(x)=-\infty$.

Looking at $f(x)=\frac{1}{x}$ (pictured her again), observe that



$\lim_{x\to0^+}\left(\frac{1}{x}\right)=\infty$ and $\lim_{x\to0^-}\left(\frac{1}{x}\right)=-\infty$.

We say that the line $x = 0$ is a ***vertical*** *asymptote* of the graph of f.

Definition: Vertical Asymptote

The line $x=b$ is a vertical asymptote of the graph of a function $f(x)$ if either $\lim_{x\to0^+} f(x)=b$, or $\lim_{x\to0^-} f(x)=b$.

Investigation 2: Given $f(x)=\frac{1}{x^2-4}$. Find the vertical asymptotes of the graph of f(x) and describe the behavior of f(x) to the left and right of each vertical asymptote.

IV. Exercises

1. Use graphs and tables to find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ and identify all horizontal asymptotes.

a) $f(x) = \cos\left(\frac{1}{x}\right)$

b) $f(x) = \frac{e^{-x}}{x}$

c) $f(x) = \frac{x}{|x|}$

d) $f(x) = \frac{3x^3 - x + 1}{x + 3}$

2. Use graphs and tables to find the limits.

a) $\lim_{x\to 2^+} \left(\frac{1}{x-2}\right)$

b) $\lim_{x\to 3^-} \left(\frac{1}{x} + 3\right)$

c) $\lim_{x\to 0^+} (\csc x)$

d) $\lim_{x\to 0^+} (\sec x)$

3. Find the vertical asymptotes of the graph of g(x) and describe the behavior of g(x) to the left and right of each vertical asymptote.

a) $g(x) = \frac{x^2 - 1}{2x + 4}$

b) $g(x) = \frac{x^2 - 2x}{x + 1}$

b) $g(x) = \frac{\tan x}{\sin x}$

IV. Assessment – Khan Academy

Complete the next five online practice exercises in the fourth unit (Infinite Limits) of Khan Academy's AP Calculus AB course:

- https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-continuity/ab-infinite-limits/e/unbounded-limits-graphical
- https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-continuity/ab-infinite-limits/e/limits-at-infinity-where-f-x--is-unbounded
- https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-continuity/ab-limits-at-infinity/e/limits-at-infinity-where-x-is-unbounded
- https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-continuity/ab-limits-at-infinity/e/limits-at-infinity-of-rational-functions-radicals
- https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-continuity/ab-limits-at-infinity/e/limits-at-infinity-of-rational-functions-trig