1.3 Limits

Calculating Limits

Functions are at the heart of mathematics: a function is a process or rule that associates each individual input to exactly one corresponding output. Students learn in courses prior to calculus that there are many different ways to represent functions, including through formulas, graphs, tables, and even words. For example, the squaring function can be thought of in any of these ways. In words, the squaring function takes any real number x and computes its square. The formulaic and graphical representations go hand in hand, as $f(x) = x^2$ is one of the simplest curves to graph. Finally, you can also partially represent this function through a table of values, essentially by listing some of the ordered pairs that lie on the curve, such as (-2, 4), (-1, 1), (0, 0), (1, 1), and (2, 4).

Functions are especially important in calculus because they often model important phenomena – the location of a moving object at a given time, the rate at which an automobile is consuming gasoline at a certain velocity, the reaction of a patient to the size of a dose of a drug – and calculus can be used to study how these output quantities change in response to changes in the input variable. Moreover, thinking about concepts like average and instantaneous velocity leads you naturally from an initial function to a related, sometimes more complicated function. As one example of this, think about the falling ball whose position function is given by $s(t) = 64 - 16t^2$ and the average velocity of the ball on the interval [1, x]. Observe that:

$$AV_{[1,x]} = \frac{s(x) - s(1)}{x - 1} = \frac{(64 - 16x^2) - (64 - 16)}{x - 1} = \frac{16 - 16x^2}{x - 1}$$

Now, two things are essential to note: this average velocity depends on x (indeed, $AV_{[1,x]}$ is a function of x), and your most focused interest in this function occurs near $x = 1$, which is where the function is not defined. Said differently, the function $g(x) = \frac{16-16x^2}{x-1}$ tells you the average velocity of the ball on the interval from $t = 1$ to t = x, and if you are interested in the instantaneous velocity of the ball when $t = 1$, you'd like to know what happens to $g(x)$ as x gets closer and closer to 1. At the same time, $g(1)$ is not defined, because it leads to the quotient 0/0.

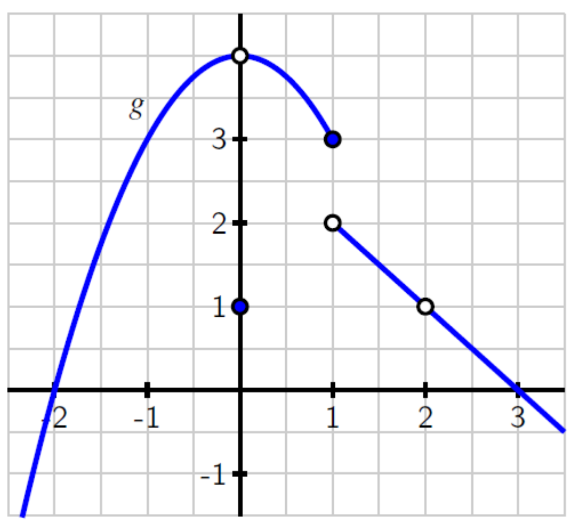
This is where the idea of limits comes in. By using a limit, you'll be able to allow x to get arbitrarily close, but not equal, to 1 and fully understand the behavior of $g(x)$ near this value. You will develop key language, notation, and conceptual understanding in what follows, but for now consider a preliminary investigation that uses the graphical interpretation of a function to explore points on a graph where interesting behavior occurs.

Investigation 1: Suppose that g is the function given by the graph below. Use the graph to answer each of the following questions.

1a) Determine the values $g(-2), g(-1), g(0), g(1), g(2)$, if defined. If the function value is not defined, explain what feature of the graph tells you this.

b) For each of the values $a =$ -1, $a =$ 0, and $a =$ 2, complete the following sentence: "As x gets closer and closer (but not equal) to a, $g(x)$ gets as close as you want to _____________________________."

c) What happens as x gets closer and closer (but not equal) to $a = 1$? Does the function $g(x)$ get as close as you would like to a single value?



II. The Notion of Limit

Limits can be thought of as a way to study the tendency or trend of a function as the input variable approaches a fixed value, or even as the input variable increases or decreases without bound. You will put off the study of the latter idea until further along in the course when you will have some helpful calculus tools for understanding the end behavior of functions. Here, just focus on what it means to say that "a function f has limit L as x approaches a." To begin, think about a recent example.

In Investigation 1, you saw that for the given function g, as x gets closer and closer (but not equal) to 0, $g(x)$ gets as close as you want to the value 4. At first, this may feel counterintuitive, because the value of $g(0)$ is 1, not 4. By their very definition, limits regard the behavior of a function *arbitrarily close* to a fixed input, but the value of the function at the fixed input does not matter. More formally[1], you say the following.

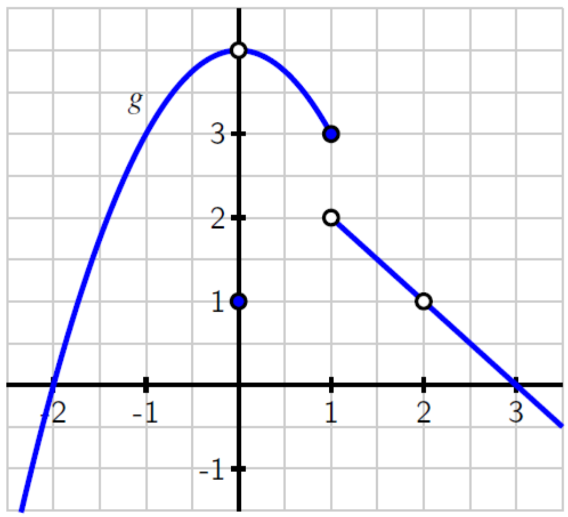
> **Definition:** Given a function f, a fixed input $x = a$, and a real number L, we say that
>
> f has limit L as x approaches a, and write
>
> $$\lim_{x \to a} f(x) = L\,,$$
>
> provided that you can make $f(x)$ as close to L as you like by taking x sufficiently close (but not equal) to a. If you cannot make $f(x)$ as close to a single value as you would like as x approaches a, then you say that f does not have a limit as x approaches a.

[1] What follows here is not what mathematicians consider the formal definition of a limit. To be completely precise, it is necessary to quantify both what it means to say "as close to L as you like" and "sufficiently close to a." That can be accomplished through what is traditionally called the epsilon-delta definition of limits. The definition presented here is sufficient for the purposes of this text.

For the function g pictured below, you can make the following observations:

$$\lim_{x \to -1} g(x) = 3, \; \lim_{x \to 0} g(x) = 4 \; and \; \lim_{x \to 2} g(x) = 1,$$

but g does not have a limit as $x \to 1$. When working graphically, it suffices to ask if the function approaches a single value from each side of the fixed input, while understanding that the function value right at the fixed input is irrelevant. This reasoning explains the values of the first three stated limits. In a situation such as the jump in the graph of g at x = 1, the issue is that if you approach x = 1 from the left, the function values tend to get as close to 3 as you'd like, but if you approach x = 1 from the right, the function values get as close to 2 as you'd like, and there is no single number that all of these function values approach. Therefore, the limit of g does not exist at x = 1.



For any function f , there are typically three ways to answer the question "does f have a limit at x = a, and if so, what is the limit?" The first is to reason graphically as you have just done with the example from Investigation 1. If you have a formula for $f(x)$, there are two additional possibilities: (1) evaluate the function at a sequence of inputs that approach a on either side, typically using some sort of computing technology, and ask if the sequence of outputs seems to approach a single value; (2) use the algebraic form of the function to understand the trend in its output as the input values approach a. The first approach only produces an approximation of the value of the limit, while the latter can often be used to determine the limit exactly. The following example demonstrates both of these approaches, while also using the graphs of the respective functions to help confirm your conclusions.

Example 1. For each of the following functions, decide whether or not the function has a limit at the stated a-values. Use both numerical and algebraic approaches to investigate and, if possible, estimate or determine the value of the limit. Compare the results with a careful graph of the function on an interval containing the points of interest.

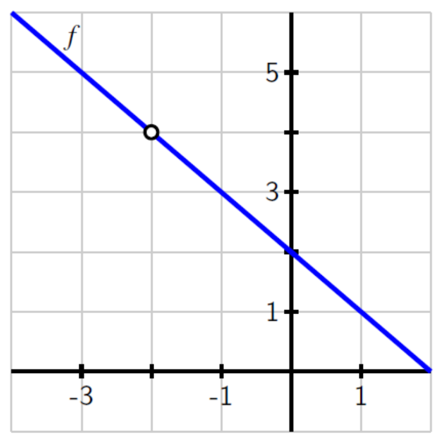
(a) $f(x) = \dfrac{4 - x^2}{x + 2}$; $a = -1$, $a = -2$

(b) $g(x) = \sin\left(\dfrac{\pi}{x}\right)$; $a = 3$, $a = 0$

Solution. You first construct a graph of f along with tables of values near a = -1 and a = -2.

x	$f(x)$
-0.9	2.9
-0.99	2.99
-0.999	2.999
-0.9999	2.9999
-1.1	3.1
-1.01	3.01
-1.001	3.001
-1.0001	3.0001

x	$f(x)$
-1.9	3.9
-1.99	3.99
-1.999	3.999
-1.9999	3.9999
-2.1	4.1
-2.01	4.01
-2.001	4.001
-2.0001	4.0001



From the left table, it appears that you can make f as close as you want to 3 by taking x sufficiently close to -1, which suggests that $\lim_{x \to -1} f(x) = 3$,. This is also consistent with the graph of f . To see this a bit more rigorously and from an algebraic point of view, consider the formula for f: $f(x) = \frac{4-x^2}{x+2}$. The numerator and denominator are each polynomial functions, which are among the most well-behaved functions that exist. Formally, such functions are continuous, which means that the limit of the function at any point is equal to its function value. Here, it follows that as $x \to -1$, $(4 - x^2) \to (4 - (-1)^2 = 3$ and $(x + 2) \to (-1 + 2) = 1$, so as $x \to -1$, the numerator of f tends to 3 and the denominator tends to 1, hence $\lim_{x \to -1} f(x) = \frac{3}{1} = 3$.

The situation is more complicated when $x \to -2$, due in part to the fact that f (-2) is not defined. If you use a similar algebraic argument regarding the numerator and denominator, you observe that as $x \to -2$, $(4 - x^2) \to (4 - (-2)^2 = 0$ and $(x + 2) \to (-2 + 2) = 0$, so as $x \to -2$, the numerator of f tends to 0 and the denominator tends to 0. You call 0/0 an ***indeterminate form*** and will revisit several important issues surrounding such quantities later. For now, simply observe that this tells you there is more work to do. From the table and the graph, it appears that f should have a limit of 4 at x = -2. To see algebraically why this is the case, let's work directly with the form of f (x). Observe that

$$\lim_{x\to-2} f(x) = \lim_{x\to-2} \frac{4 - x^2}{x + 2}$$

$$= \lim_{x\to-2} \frac{(2 - x)(2 + x)}{x + 2}$$

At this point, it is important to observe that since you are taking the limit as $x \to -2$, you are considering x values that are close, but not equal, to -2. Since you never actually allow x to equal -2, the quotient 2+x has value 1 for every possible value of x. Thus, you can simplify the most recent expression above, and now find that

$$\lim_{x\to-2} f(x) = \lim_{x\to-2} \frac{(2 - x)\cancel{(2 + x)}}{\cancel{x + 2}} = \lim_{x\to-2} 2 - x.$$

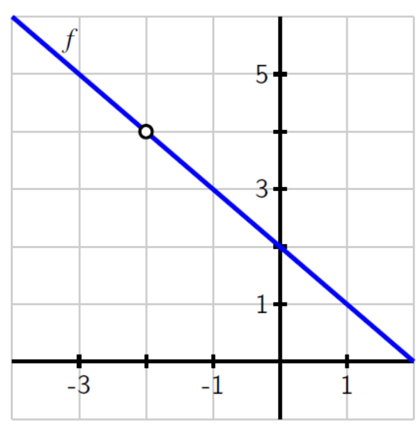
Because 2 - x is simply a linear function, this limit is now easy to determine, and its value clearly is 4. Thus, from several points of view you've seen that

$$\lim_{x\to-2} = 4$$

Next, turn to the function g, and construct two tables and a graph.

x	$f(x)$
-0.9	2.9
-0.99	2.99
-0.999	2.999
-0.9999	2.9999
-1.1	3.1
-1.01	3.01
-1.001	3.001
-1.0001	3.0001

x	$f(x)$
-1.9	3.9
-1.99	3.99
-1.999	3.999
-1.9999	3.9999
-2.1	4.1
-2.01	4.01
-2.001	4.001
-2.0001	4.0001



First, as $x \to 3$, it appears from the data (and the graph) that the function is approaching approximately 0.866025. To be precise, you have to use the fact that $\frac{\pi}{x} \to \frac{\pi}{3}$, and thus find that $g(x) = \sin\left(\frac{\pi}{x}\right) \to \sin\left(\frac{\pi}{3}\right)$ as $x \to 3$. The exact value of $\sin\left(\frac{\pi}{3}\right)$ is $\frac{\sqrt{3}}{2}$, which is approximately 0.8660254038. Thus, you see that

$$\lim_{x \to 3} g(x) = \frac{\sqrt{3}}{2}$$

As $x \to 0$, observe that $\frac{\pi}{x}$ does not behave in an elementary way. When x is positive and approaching zero, you are dividing by smaller and smaller positive values, and $\frac{\pi}{x}$ increases without bound. When x is negative and approaching zero, $\frac{\pi}{x}$ decreases without bound. As you get close to x = 0, the inputs to the sine function are growing rapidly, and this leads to wild ***oscillations*** in the graph of g. It is an instructive

x = 0. Doing so shows that the function never settles down to a single value near the origin and suggests that g does not have a limit at x = 0.

How do you reconcile this with the right-hand table above, which seems to suggest that the limit of g as x approaches 0 may in fact be 0? Here you need to recognize that the data misleads you because of the special nature of the sequence {0.1, 0.01, 0.001, . . .}: when you evaluate $g(10^{-k}) = \sin\left(\frac{\pi}{10^{-k}}\right) = \sin(10^k \pi) = 0$ for each integer value of k. But if you take a different sequence of values approaching zero, say {0.3, 0.03, 0.003, . . .}, then you find that

$$g(3 \cdot 10^{-k}) = \sin\left(\frac{\pi}{3 \cdot 10^{-k}}\right) = \sin\left(\frac{10^k \pi}{3}\right) = -\frac{\sqrt{3}}{2} \approx -0.866025.$$

That sequence of data would suggest that the value of the limit is $\frac{\sqrt{3}}{2}$. Clearly the function cannot have two different values for the limit, and this shows that g has no limit as $x \to 0$.

An important lesson to take from Example 1 is that tables can be misleading when determining the value of a limit. While a table of values is useful for investigating the possible value of a limit, you should also use other tools to confirm the value, if you think the table suggests the limit exists.

Investigation 2: Estimate the value of each of the following limits by constructing appropriate tables of values. Then determine the exact value of the limit by using algebra to simplify the function. Finally, plot each function on an appropriate interval to check your result visually.

2a) $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

b) $\lim_{x \to 0} \frac{(2+x)^3 - 8}{x}$

c) $\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$

This concludes a rather lengthy introduction to the notion of limits. It is important to remember that your primary motivation for considering limits of functions comes from your interest in studying the rate of change of a function. To that end, you close this section by revisiting your previous work with average and instantaneous velocity and highlighting the role that limits play.

III. Exercises

1. Consider the function whose formula is $f(x) = \frac{16-x^4}{x^2-4}$.

a) What is the domain of f ?

b) Use a sequence of values of x near a = 2 to estimate the value of $\lim_{x\to 2} f(x)$, if you think the limit exists. If you think the limit doesn't exist, explain why.

c) Evaluate $\lim_{x\to 2} f(x)$ exactly, if the limit exists, or explain how your work shows the limit fails to exist. Here you should use algebra to factor and simplify the numerator and denominator of f (x) as you work to evaluate the limit. Discuss how your findings compare to your results in (b).

d) True or false: f (2) = -8. Why?

e) True or false: $\frac{16-x^4}{x^2-4} = -4 - x^2$. Why? How is this equality connected to your work above with the function f ?

f) Based on all of your work above, construct an accurate, labeled graph of y = f (x) on the interval [1, 3], and write a sentence that explains what you now know about

$$\lim_{x\to 2} \frac{16 - x^4}{x^2 - 4}.$$

2. Let $g(x) = \frac{|x+3|}{x+3}$.

a) What is the domain of g?

b) Use a sequence of values near a = 3 to estimate the value of $\lim_{x\to -3} g(x)$, if you think the limit exists. If you think the limit doesn't exist, explain why.

c) Evaluate $\lim_{x\to -3} g(x)$ exactly, if the limit exists, or explain how your work shows the limit fails to exist. Here you should use the definition of the absolute value function in the numerator of g(x) as you work to evaluate the limit. Discuss how your findings compare to your results in (b). (**Hint**: |a| = a whenever a > 0, but |a| = -a whenever a < 0.)

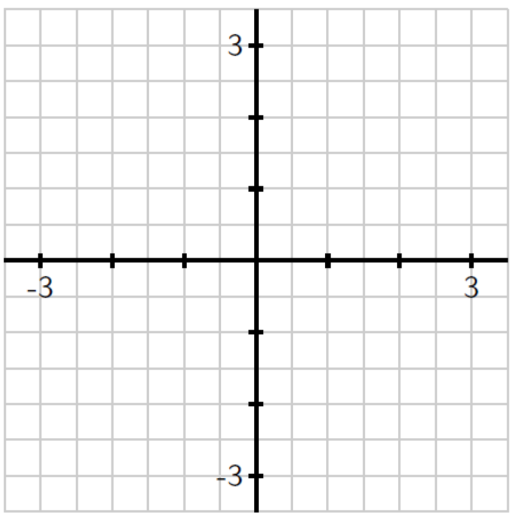
d) True or false: g(-3) = -1. Why?

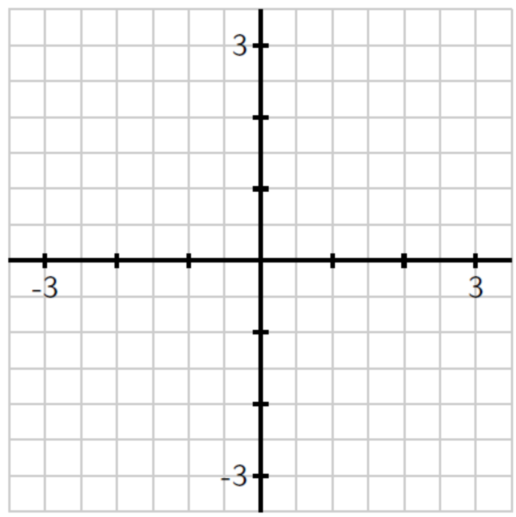
e) True or false: |x+3| = 1. Why? How is this equality connected to your work above with the function g?

f) Based on all of your work above, construct an accurate, labeled graph of

y = g(x) on the interval [-4, -2], and write a sentence that explains what you now know about $\lim_{x \to -3} g(x)$.

3. For each of the following prompts, sketch a graph on the provided axes of a function that has the stated properties.





a) y = f (x) such that

- f(-2) = 2 and $\lim_{x \to -2} f(x) = 1$
- f(-1) = 3 and $\lim_{x \to -1} f(x) = 3$
- f(1) is not defined and $\lim_{x \to 1} f(x) = 0$
- f(2) = 1 and $\lim_{x \to -2} f(x)$ does not exist.

b) y = g(x) such that

- g(-2) = 3, g(-1) = -1, g(1) = -2, and g(2) = 3
- At x = 2, 1, 1 and 2, g has a limit, and its limit equals the value of the function at that point.
- g(0) is not defined and $\lim_{x \to 0} g(x)$ does not exist.

IV. Practice – Khan Academy

Complete the next seven online practice exercises in the first unit (Limits and Continuity) of Khan Academy's AP Calculus AB course:

- https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-continuity/ab-direct-sub/e/find-limits-by-direct-substitution
- https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-continuity/ab-direct-sub/e/identify-undefined-limits-by-direct-substitution
- https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-continuity/ab-factor-and-ratio/e/two-sided-limits-using-algebra
- https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-continuity/ab-factor-and-ratio/e/limits_2
- https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-continuity/ab-trig-and-squeeze/e/determining-trigonometric-limits
- https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-continuity/ab-trig-and-squeeze/e/find-limits-using-trig-identities
- https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-continuity/ab-trig-and-squeeze/e/squeeze-theorem

Extra: First two Khan quizzes