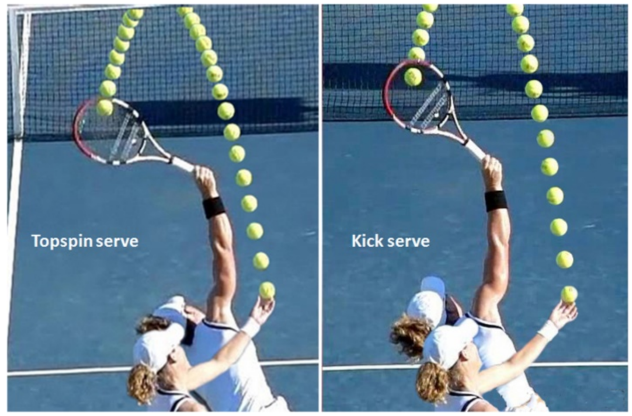
1.1 Advantage-In

Rates of Change and Tangent Lines



Calculus can be viewed broadly as the study of change. A natural and important question to ask about any changing quantity is "how fast is the quantity changing?" It turns out that in order to make the answer to this question precise, substantial mathematics is required.

You will begin with a familiar problem: a ball being tossed straight up in the air from an initial height.

Investigation 1: Suppose that the height s of a ball (in feet) at time t (in seconds) is

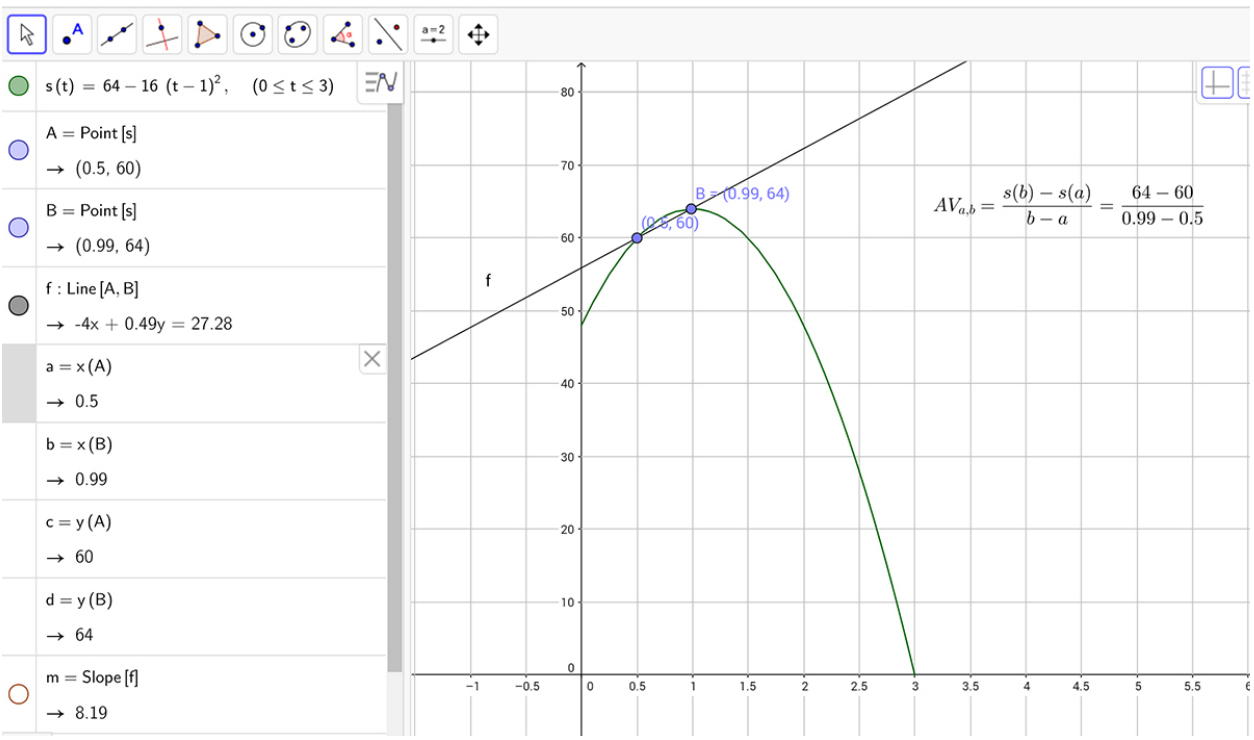
given by the formula $s(t) = 64 - 16(t-1)^2$

1. Draw an accurate graph of $y = s(t)$ on the time interval $0 \leq t \leq 3$. Label at least six distinct points on the graph, including the three points that correspond to when the ball was released, when the ball reaches its highest point, and when the ball lands. Sketch and label a copy of your graph below.

2. In everyday language, describe the behavior of the ball on the time interval $0 < t < 1$ and on time interval $1 < t < 3$. What occurs at the instant $t = 1$?

3. Use GeoGebra to plot the graph of $s(t) = 64 - 16(t-1)^2$ on the same interval you used in question 1.

- Restrict the domain to $0 \le t \le 3$. (see top left entry in image below)
- Make sure points A and B can move freely (may need to rename the points)
    - o Change Point button to Point on Object
- Construct a line from A to B (make sure both points can still move)
- Define:
    - o $a = x(A)$ (variable a is defined as the x-value of point A)
    - o $b = x(B)$ (variable b is defined as the x-value of point B)
    - o $c = y(A)$ (variable c is defined as the x-value of point A)
    - o $d = y(B)$ (variable d is defined as the x-value of point B)
- Create a formula (text button) and use Latex Code



4. Use the graph to compute the average velocity of the ball on each of the following time intervals: (Use a calculator to complete the calculation, if necessary and include units for each value.)

a) $AV_{[0.5,1]}$ = ---------------------- =

b) $AV_{[1,2]}$ =

c) $AV_{[2,3]}$ =

3. Consider the expression $AV_{[0.5,1]} = \frac{s(1)-s(0.5)}{1-0.5}$. Compute the value of $AV_{[0.5,1]}$.

a) What does this value measure geometrically? (Hint: one of the trig functions.)

b) What does this value measure physically?

c) What are the units on $AV_{[0.5,1]}$.

II. Position and average velocity

Any moving object has a ***position*** that can be considered a function of *time*. When this motion is along a straight line, the position is given by a single variable, and you usually let this position be denoted by $s(t)$, which reflects the fact that position is a function of time. For example, you might view $s(t)$ as telling the mile marker of a car traveling on a straight highway at time t in hours; similarly, the function s described in Investigation 1 is a position function, where position is measured vertically relative to the ground.

Not only does such a moving object have a position associated with its motion, but on any time interval, the object has an ***average velocity***. Think, for example, about driving from one location to another: the vehicle travels some number of miles over a certain time interval (measured in hours), from which you can compute the vehicle's average velocity. In this situation, average velocity is the number of miles traveled divided by the time elapsed, which of course is given in *miles per hour*. Similarly, the calculation of $AV_{[0.5,1]}$ in Investigation 1 found the average velocity of the ball on the time interval [0.5, 1], measured in feet per second.

In general, use the following definition: for an object moving in a straight line whose position at time t is given by the function $s(t)$, the *average velocity of the object on the interval from* $t = a$ *to* $t = b$, denoted $AV_{[a,b]}$, is given by the formula

$$AV_{[a,b]} = \frac{s(b) - s(a)}{b - a}$$

Note: the units on $AV_{[a,b]}$ are "units of s per unit of t," such as "miles per hour" or "feet per second."

4. Compute the average velocity of the ball on each of the following time intervals: (Include units for each value.)

a) [0.4, 0.8]

b) [0.7, 0.8]

c) [0.79, 0.8]

d) [0.799, 0.8]

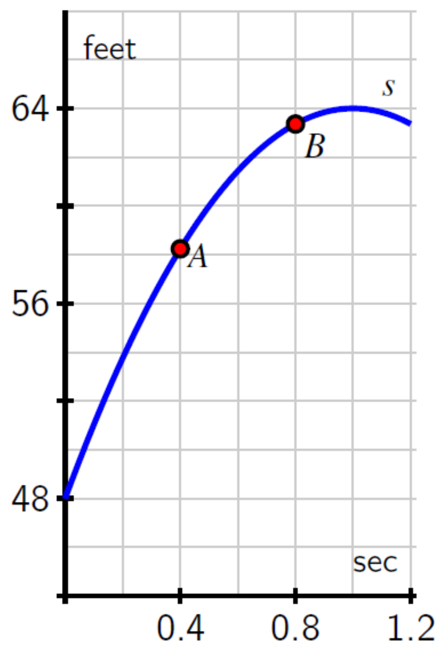
e) [0.8, 1.2]

f) [0.8, 0.9]

g) [0.8, 0.81]

h) [0,8, 0.801]

5. On the graph below, sketch the line that passes through the points $A = (0.4, s(0.4)$ and $B = (0.8, s(0.8)$. What is the meaning of the slope of this line? Considering this meaning, what is a geometric way to interpret each of the values computed in the preceding question?



III. Instantaneous Velocity

Whether driving a car, riding a bike, or throwing a ball, you have an intuitive sense that any moving object has a velocity at any given moment – a number that measures how fast the object is moving *right now*.

For instance, a car's speedometer tells the driver what appears to be the car's velocity at any given instant. In fact, the posted velocity on a speedometer is really an average velocity that is computed over a very small time interval (by computing how many revolutions the tires have undergone to compute distance traveled), since velocity fundamentally comes from considering a change in position divided by a change in time. But if you let the time interval over which average velocity is computed become shorter and shorter, then you can progress from average velocity to ***instantaneous velocity***.

Informally, you define the *instantaneous velocity* of a moving object at time $t = a$ to be the value that the average velocity approaches as you take smaller and smaller intervals of time containing $t = a$ to compute the average velocity. You will develop a more formal definition of this momentarily, one that will end up being the foundation of much of our work in first semester calculus. For now, it is fine to think of instantaneous velocity this way: take average velocities on smaller and smaller time intervals, and if those average velocities approach a single number, then that number will be the instantaneous velocity at that point.

Investigation 6: Use GeoGebra to plot the graph of $s(t) = 64 - 16(t-1)^2$ on the same interval you used in Investigation 1.

- Construct two points on the curve that move freely
- Construct a line that connects the two points

6a) Zoom in repeatedly on the point $(0.8, s(0.8))$. What do you observe about how the graph appears as you view it more and more closely?

b) What do you conjecture is the velocity of the ball at the instant $t = 0{:}8$? Why?

c) Compute the average velocity of the ball on the time interval [1.5, 2]. What is different between this value and the average velocity on the interval [0, 0.5]?

d) Use appropriate computing technology to estimate the instantaneous velocity of the ball at $t = 1{:}5$. Likewise, estimate the instantaneous velocity of the ball at $t = 2$. Which value is greater?

e) How is the sign of the instantaneous velocity of the ball related to its behavior at a given point in time? That is, what does positive instantaneous velocity tell you the ball is doing? Negative instantaneous velocity?

f) Without doing any computations, what do you expect to be the instantaneous velocity of the ball at $t = 1$? Why?

If you desire to know the instantaneous velocity at $t = a$ of a moving object with position function s, you are interested in computing average velocities on the interval $[a,b]$ for smaller and smaller intervals. One way to visualize this is to think of the value b as being $b = a + h$, where h is a small number that is allowed to vary. Thus, you observe that the average velocity of the object on the interval $[a,a+h]$ is

$$AV_{[a,a+h]} = \frac{s(a+h) - s(a)}{h}$$

with the denominator being simply h because $(a + h) - a = h$. Initially, it is fine to think of h being a small positive real number; but it is important to note that you allow h to be a small negative number, too, as this enables you to investigate the average velocity of the moving object on intervals prior to $t = a$, as well as following $t = a$. When $h < 0$, $AV_{[a,a+h]}$ measures the average velocity on the interval $[a + h, a]$.

To attempt to find the instantaneous velocity at $t = a$, you investigate what happens as the value of h approaches zero. Consider this further in the following example.

Example 1. For a falling ball whose position function is given by $s(t) = 16 - 16t^2$ (where s is measured in feet and t in seconds), find an expression for the average velocity of the ball on a time interval of the form $[0.5, 0.5 + h]$ where $-0.5 < h < 0.5$ and $h \neq 0$. Use this expression to compute the average velocity on [0.5, 0.75] and [0.4, 0:5], as well as to make a conjecture about the instantaneous velocity at $t = 0{:}5$.

Solution. You make the assumptions that -0.5 < h < 0.5 and $h \neq 0$ because h cannot be zero (otherwise there is no interval on which to compute average velocity) and because the function only makes sense on the time interval $0 \leq t \leq 1$, as this is the duration of time during which the ball is falling. Observe that you want to compute and simplify

$$AV_{[0.5,0.5+h]} = \frac{s(0.5+h) - s(0.5)}{(0.5+h) - 0.5}$$

The most unusual part of this computation is finding $s(0.5 + h)$. To do so, you follow the rule that defines the function s. In particular, since $s(t) = 16 - 16t^2$, therefore

$$s(0.5 + h) = 16 - 16(0.5 + h)^2$$

$$= 16 - 16(0.25 + h + h^2)$$

$$= 16 - 4 - 16h - 16h^2$$

$$= 12 - 16h - 16h^2.$$

Now, returning to your computation of the average velocity, you find that

$$AV_{[0.5,0.5+h]} = \frac{s(0.5+h) - s(0.5)}{(0.5+h) - 0.5}$$

$$= \frac{(12 - 16h - 16h^2) - (16 - 16(0.5)^2)}{(0.5+h) - 0.5}$$

$$= \frac{12 - 16h - 16h^2 - 12}{h}$$

$$= \frac{16h - 16h^2}{h}$$

Since h can never equal zero, you may further simplify the most recent expression. Removing the common factor of h from the numerator and denominator, it follows that

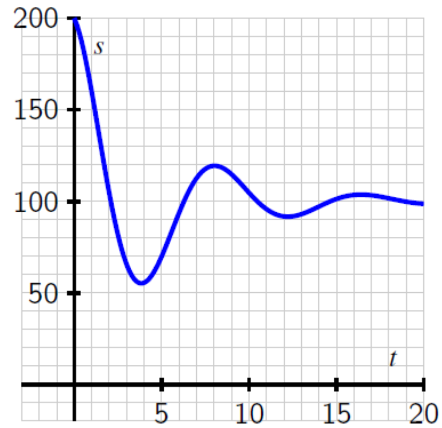
$$AV_{[0.5,0.5+h]} = -16 - 16h.$$

Now, you can even explore what happens to $AV_{[0.5,0.5+h]}$ as h gets closer and closer to zero. As h approaches zero, - $16h$ will also approach zero, and thus it appears that the instantaneous velocity of the ball at $t = 0.5$ should be -16 ft/sec.

Investigation 6: For the function given by $s(t) = 64 - 16(t-1)^2$ from Investigation 1, find the most simplified expression you can for the average velocity of the ball on the interval $[2, 2+h]$. Use your result to compute the average velocity on 1.5, 2 and to estimate the instantaneous velocity at t = 2.

IV. Exercises

1. A bungee jumper dives from a tower at time $t = 0$. Her height h (measured in feet) at time t (in seconds) is given by the graph at the right.



In this problem, you may base your answers on estimates from the graph or use the fact that the jumper's height function is given by $s(t) = 100\cos(0.75t) \cdot e^{-0.2t} + 100$.

a) What is the change in vertical position of the bungee jumper between $t = 0$ and $t = 15$?

b) Estimate the jumper's average velocity on each of the following time intervals: [0, 15], [0, 2], [1, 6], and [8, 10]. Include units on your answers.

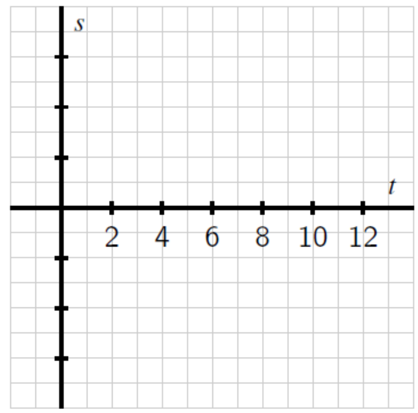
c) On what time interval(s) do you think the bungee jumper achieves her greatest average velocity? Why?

d) Estimate the jumper's instantaneous velocity at $t = 5$. Show your work and explain your reasoning, and include units on your answer.

e) Among the average and instantaneous velocities you computed in earlier questions, which are positive and which are negative? What does negative velocity indicate?

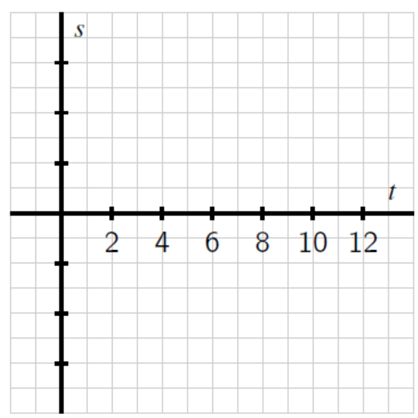
2. A diver leaps from a 3 meter springboard. His feet leave the board at time $t = 0$, he reaches his maximum height of 4.5 m at t = 1.1 seconds, and enters the water at $t = 2.45$. Once in the water, the diver coasts to the bottom of the pool (depth 3.5 m), touches bottom at $t = 7$, rests for one second, and then pushes off the bottom. From there he coasts to the surface, and takes his first breath at $t = 13$.

a) Let $s(t)$ denote the function that gives the height of the diver's feet (in meters) above the water at time t. (Note that the "height" of the bottom of the pool is -3.5 meters.) Sketch a carefully labeled graph of $s(t)$ on the provided axes. Include scale and units on the vertical axis. Be as detailed as possible.



b) Based on your graph in (a), what is the average velocity of the diver between $t = 2.45$ and $t = 7$? Is his average velocity the same on every time interval within [2.45, 7]?

c) Let the function $v(t)$ represent the instantaneous vertical velocity of the diver at time t (i.e. the speed at which the height function $s(t)$ is changing; note that velocity in the upward direction is positive, while the velocity of a falling object is negative). Based on your understanding of the diver's behavior, as well as your graph of the position function, sketch a carefully labeled graph of $v(t)$ on the axes provided. Include scale and units on the vertical axis. Write several sentences that explain how you constructed your graph, discussing when you expect $v(t)$ to be zero, positive, negative, relatively large, and relatively small.



d) Is there a connection between the two graphs that you can describe? What can you say about the velocity graph when the height function is increasing? decreasing? Make as many observations as you can.

3. According to the U.S. census, the population of the city of Grand Rapids, MI, was 181,843 in 1980; 189,126 in 1990; and 197,800 in 2000.

a) Between 1980 and 2000, by how many people did the population of Grand Rapids grow?

b) In an average year between 1980 and 2000, by how many people did the population of Grand Rapids grow?

c) Just like you can find the average velocity of a moving body by computing change in position over change in time, you can compute the ***average rate of change*** of any function f . In particular, the average rate of change of a function f over an interval [a, b] is the quotient

$$\frac{f(b) - f(a)}{b - a}$$

What does the quantity $f(b)$- $f(a)$ measure on the graph of $y = f(x)$ over the interval $[a, b]$?

d) Let $P(t)$ represent the population of Grand Rapids at time t, where t is measured in years from January 1, 1980. What is the average rate of change of P on the interval $t = 0$ to $t = 20$? What are the units on this quantity?

e) If you assume the population of Grand Rapids is growing at a rate of approximately 4% per decade, you can model the population function with the formula

$$P(t) = 181843(1.04)^{t/10}$$

Use this formula to compute the average rate of change of the population on the intervals [5, 10], [5, 9], [5, 8], [5, 7], and [5, 6].

f) How fast do you think the population of Grand Rapids was changing on January 1, 1985? Said differently, at what rate do you think people were being added to the population of Grand Rapids as of January 1, 1985? How many additional people should the city have expected in the following year? Why?

III. Assessment – Khan Academy

1. https://www.khanacademy.org/math/ap-calculus-ab/ab-derivative-intro/ab-estimate-derivatives/e/secants-and-average-rate-of-change