Activity 5.3: Beyond the Linear

In this activity you explore a what-if situation that involves some simple assumptions. These assumptions contain features that will be used in a reproduction model. When you work with them, a new type of growth pattern will be encountered, which is not an additive process. You learn ways to describe that pattern of growth in mathematical terms.

Building a reproduction model will require new mathematical tools to describe the patterns that are produced. For that reason, you will get away from the actual moose problem in this lesson, taking a different approach to modeling. As a way of exploring those patterns, and learning how to describe them, you will play "what if…." This will give you some data, which you can then analyze.

To generate the data, begin with a simple set of assumptions. What if:

- The starting population is 16 moose, 8 of them females and 8 males.
- Every female has a baby each year.
- Exactly half of the babies each year are females.
- Every baby female has its own babies the next year.

This scenario is much too simple to be realistic. The second step of the modeling process suggests that you start simply. You will find that these assumptions and values dictate what happens from year to year. Everybody will get the same results as long as you follow the assumptions. For that reason this is called a **deterministic simulation**, a type of role-play in which the outcome is uniquely determined by the conditions.

MODELING NOTE

For now, the focus is going to be on tracking the size of the entire herd, even though the population is being determined by assumptions that apply to individual moose.

You will walk through a few years of population growth. That will produce enough data for you to be able to find the underlying pattern.

1. Make and complete a data table like the one shown in **Figure 5.20** to keep track of the results of this simulation.

Year number (t)	No. of females (F)	No. of males (M)	Newborn females (ΔF)	Newborn males (ΔM)	Total population (P)
0	–	–	–	–	16
1	8	8			
2					
3					

Figure 5.20.

2. Is the population growth an additive process? Explain.

3. In completing the table, you worked with many relationships. For each of the following, write an equation to describe how the quantities are related.

   a) This year's total population and the number of males and females at the beginning of next year.

   b) The number of newborn females (ΔF) and the number of females at the beginning of that same year.

   c) The change in the total population during a year and the number of females there were at the beginning of that year.

   d) The change in the total population during a year and the total population at the end of the previous year.

TAKE NOTE

Don't forget about arrow diagrams! They are useful tools for thinking about what the relationships are, and guiding you through writing them as an equation!

4. The organization of the data table was necessary for the simulation, but is somewhat cluttered. Make a new data table that only keeps track of the following quantities: the population for the previous year (P_{previous}), the population for the current year (P_{current}), and the change in population that occurs during that year (ΔP).

   a) Explain where the numbers came from that fill the first row of your new table.

   b) What percent of the previous population (P_{previous}) is the amount of change in the population (ΔP)? In answering that question, does it matter which year you use?

   c) Write an equation that calculates the change in population (ΔP) from the previous population (P_{previous}). Also, write an equation to calculate the current population (P_{current}) for any year from the other entries in your table for that year.

   d) Now use your answers from part (c) to write a recursive equation that calculates the current moose population (P_{current}) in terms of the previous population (P_{previous}) only.

MODELING NOTE

Some of the equations just developed use previous values of one quantity to determine present values of *another* quantity. This is different from recursive descriptions that you have seen so far, and is a powerful modeling tool.

In Chapter 4, *Animation*, the recursive process describing the pixel location was summarized by the relationship:

current amount = previous amount + amount of change

That process created a new value from an old one by including some change based on the motion. Each recursive step was over a fixed time interval, and the rate of motion was assumed to be constant. So, the amount of change became a constant in the recursive equation. Adding a constant created the change, so it was called an additive process. The migration model, developed earlier in this chapter, is another example of this additive process. The number of moose that migrated into the park determined the change in population. Those moose were included in the new population totals by adding them to the previous totals. Since the migration rate was assumed constant, the amount of change ended up being constant.

The growth process in this activity does *not* have a constant amount of change for equal time intervals. In fact, you found that it is a certain percent of the previous amount per time period. As a result, this kind of growth can be characterized as a recursive equation in the following general way:

$$A_{\text{current}} = A_{\text{previous}} + r \times A_{\text{previous}}$$

where A_{previous} and A_{current} are amounts at the beginning and end of a time period, and r is the percent of increase (or decrease) per time period expressed as a decimal.

> The constant r in the recursive equation
>
> $A_{\text{current}} = A_{\text{previous}} + r \times A_{\text{previous}}$
>
> is called the **relative rate of change** (or the relative rate of growth) for a given period of time. The units for r are "percent per period."

5. Examine the connection between the population and the amount of change in the population more closely.

   a) Make a scatter plot graph of ΔP versus P_{previous} for the data in your table from Question 4. What does the spacing between the plotted points reveal about the data?

   b) What is the equation of the line that goes through the data points?

   c) How can the general equation, $A_{\text{current}} = A_{\text{previous}} + r \times A_{\text{previous}}$, be modified so that it describes your answer to part (b)? (Hint: how do you calculate the change in population, ΔDP, during some time period?)

MODELING NOTE

Modelers check whether the change in the amount of a quantity is proportional to the amount that you have. That "test" can be used to identify this kind of growth pattern.

6. Until now you have focused on how the previous year's population affects the change in the population. Now see how the previous year's and the current year's populations are related. Add another column to your data table from Question 4. Label it "Ratio."

   a) For each line in the data table, calculate the ratio between P_{current} and P_{previous} (in other words, find $P_{\text{current}}/P_{\text{previous}}$) and record it in the new column. What do you notice about these entries?

   b) What percent of the previous population is the current population? (Hint: What percent of the previous population is the *previous* population?)

c) Use the results from part (a) to write a recursive equation that predicts $P_{current}$ from $P_{previous}$. How does this equation differ from your answer to Question 5(b))?

d) Compare the value of the ratio $P_{current}/P_{previous}$ (from part (a)) with the value of the relative rate of change, r (from Question 4(b)). For this problem, what is the relationship between those two numbers?

This latest discovery may not seem much different from what was already known. However, the recursive equation had been two steps—calculating the percent of increase or decrease, and then combining that with the previous amount. Now, it is possible to skip finding the change that a quantity undergoes, and calculate the next amount directly. Also, that calculation requires one step—multiplying by a number.

The recursive process at the core of this growth can be described in the following relationship:

$$\text{current amount} = \text{control number} \times \text{previous amount.}$$

This kind of growth is called a **multiplicative** process (or model) because the basic calculation used to go from one amount to the next is a multiplication operation. Just like before, this calculation can be characterized in a general way by the recursive equation:

$$A_{current} = b \times A_{previous},$$

where b is the multiplier that calculates the next amount from the current amount.

> The constant b in the recursive equation
>
> $A_{current} = b \times A_{previous}$
>
> is called the **growth factor.**

In Chapter 1, *Secret Codes*, you learned algebraic tools for simplifying expressions, such as the distributive property and combining "like" terms. Those skills can be used to draw a connection between the two recursive descriptions that have been developed.

7. The general equation $A_{current} = A_{previous} + r \times A_{previous}$ has two terms that both contain the quantity $A_{previous}$.

a) The constant r is the coefficient for the second term in the equation that contains $A_{previous}$. What is the coefficient for the first term containing $A_{previous}$? Explain.

b) Re-write the equation by combining "like" terms. How does that help explain your answer to Question 6(c)?

c) Compare your answer to part (*b*) with the general equation $A_{current} = b \times A_{previous}$. For equal time intervals, what is the relationship between the growth factor, *b*, and the relative rate of change, *r*?

8. A visual representation of this multiplicative growth pattern is worth the time that it takes to examine it.

   a) Make a scatterplot graph of $P_{current}$ versus $P_{previous}$ for the data in your table from Question 4. What is the equation for the line that goes through your data points?

   b) How does this graph compare to the one from Question 5 of ΔP versus $P_{previous}$?

   c) How does the recursive graph of a multiplicative process differ from that of an additive process? (Hint: You may want to explore several additive and multiplicative growth patterns before answering the question.)

MODELING NOTE

Another modeling trick is to see if the recursive graph of a growth pattern is of the form $y = mx$. If so, then the growth is multiplicative, and the slope is the growth factor! (Warning: To use this trick, the recursive steps have to be the same size!)

You have found a basic mathematical process to describe your new key feature for a reproduction model. Its simplicity is appealing, and lends itself nicely to home-screen iteration. However, just like with additive processes, you will not want to go through twenty calculations (or even twenty pushes of a button) to project answers well into the future. To fully explore the reproduction model, you will want to be able to work with a closed-form equation as well.

9. Start with the initial population of 16 moose.

   a) Make an arrow diagram to represent each of the three years of simulation data with which you have been working. Be sure to indicate what operation you are doing at each step. Under the end of each arrow, write the population total at that time.

   b) Use your arrow diagram to explain where the population of 54 moose came from.

   c) Use mathematical notation to write that same description more compactly. Use your calculator to verify that the new expression gives the same answer of 54.

   d) Write a closed-form equation that uses the number of years (n) to predict the size of the population (P). Use your calculator to verify that $P(0)$, $P(1)$, $P(2)$, and $P(3)$ all give the same values as your data.

The two numbers that control this pattern of growth are the starting population and the number by which you are multiplying, the growth factor. The power in using this closed-form equation is that the exponent indicates how many times you want to do that multiplication. It is easier to write, and for a specific number of years, the calculator can get the answer much quicker.

TAKE NOTE When evaluating an expression like this, the order of operations requires you to do the exponentiation (b^x) first, then multiply the answer by a.

As a general rule, the number of years dictates what answer you will get, and also specifies how many times to do the multiplication. Therefore, the variable *is* the exponent, and is characterized by the following equation form:

$$y = a \cdot b^x$$

where x and y are the independent and dependent variables, a the starting amount, and b the growth factor.

When an equation has the independent variable as an exponent, it is called an **exponential equation**. Any quantity, like y, that can be described with an equation of this form is said to grow exponentially. The basic closed-form for an exponential equation is $y = a \cdot b^x$. The value b is called the **base** for the exponentiation process. Exponential equations express in closed-form what multiplicative growth processes say recursively. For that reason, multiplicative growth may only take on specific values, based on the time intervals.

10. Use the closed-form equation to predict the population at the end of the fourth year. What problem does that create with the simulation?
11. Look over the assumptions that were used in this simulation.
    a) Are there any that seem unrealistic to you? Explain.
    b) Suggest ways to make the simulation more realistic.

Activity Summary

In this activity, you:

- analyzed simulation data for the patterns produced by growth under an assumption of reproduction.
- developed two different ways to describe that pattern recursively. One of them used the relative rate, and the other used the growth factor.
- developed a closed-form exponential equation.

DISCUSSION/REFLECTION

1. How are the two numbers that control multiplicative growth used in the recursive and closed-form equations?
2. What is the difference between the growth factor and the relative rate?
3. What is the numeric relationship between the growth factor and the relative rate?