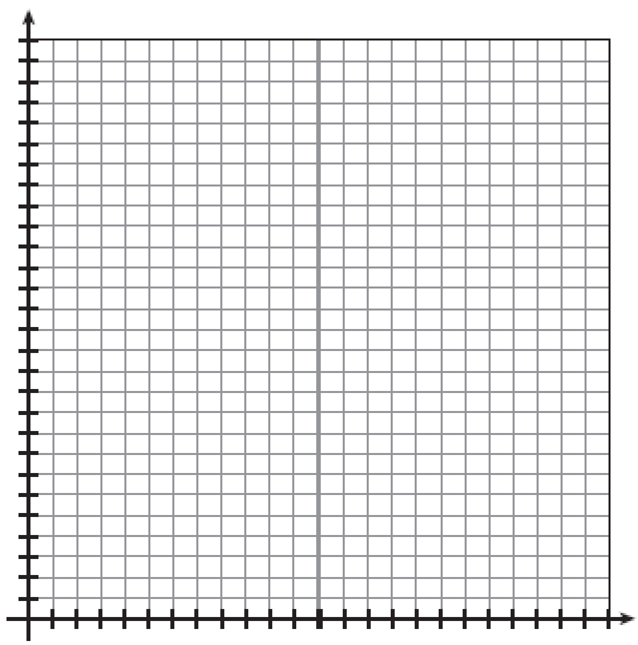
9.2: Close Enough

Limits

Magdelena is out in the middle of the lake when her boat, the *Katie Ann*, breaks down. Two friends, Alex and Briana, who live on opposite sides of the lake, receive her calls for help and agree to come to her rescue.

1. Alex's movement can be described by the parametric equations $x = t$ and $y = \frac{1}{2}t + 4$. Briana's movement can be described by the parametric equations $x = -\frac{1}{2}t + 3$ and $y = -t + 7$. In both cases, t represents hours traveled.

   a. Plot the starting location for each of Magdelena's friends on the grid below.



   b. Both friends travel half the distance to Magdelena in 1 hour. Determine the coordinates for the location of each friend after 1 hour. Plot these coordinates on the grid and trace each friend's path.

   c. From their locations after 1 hour, Alex and Briana use an additional half hour to travel half the remaining distance to Magdelena. Use $t = 1.5$ to determine the coordinates for the location of each friend after 1.5 hours. Plot these coordinates on the grid and trace each friend's path.

d. Again, both friends travel half the remaining distance to Magdelena. This time the friends travel an additional quarter of an hour. Use $t = 1.75$ to determine the coordinates for the location of each friend. Plot these coordinates on the grid and trace each friend's path.
e. A fourth time, Alex and Briana travel half the remaining distance to Magdelena, using an additional one eighth of an hour. Use $t = 1.875$ to determine the coordinates for the location of each friend. Plot these coordinates on the grid and trace each friend's path.
f. Predict the coordinates for the location of Magdelena and her boat, the *Katie Ann*. Explain your reasoning.
g. Assuming Magdelena's friends continue to move half the distance from their locations to her boat, will they ever arrive at her exact location? Explain your reasoning.

II. Limits

The concept represented by Magdelena and her friends is the idea of a limit. If an object is approached from both sides, it is possible to eventually be close enough to the object to give its exact location.

In mathematics, it is common to discuss the *limit of a function*. In order to understand the concept of a limit, you will begin by using several methods to examine functions near a certain point.

2. Let $f(x) = \frac{x^2-1}{x-1}$. Use the following methods to explore the values of y as the value of x approaches 1.
   a. Complete the table below.

x approaches 1 from the left → ← *x approaches 1 from the right*

x	**0.9**	**0.99**	**0.999**	**1**	**1.001**	**1.01**	**1.1**
f(x)							

   b. Use a graphing calculator to graph $f(x)$. Use the trace feature of your calculator to examine the values of y as the value of x approaches 1 from both directions. What happens to the values of y as the value of x approaches 1?

   c. What seems to be true about the values of y for $f(x)$ as the values of x approach 1? Do both representations of the function (table and graph) lead you to the same conclusion?

3. Let $g(x) = x^2 + 2$.
   a. Complete the table below.

x approaches 1 from the left → ← *x approaches 1 from the right*

x	0.9	0.99	0.999	1	1.001	1.01	1.1
f(x)							

   a. Use a graphing calculator to graph $g\ (x)$. Use the trace feature of your calculator to examine the values of y as the value of x approaches 1 from both directions. What happens to the values of y as the value of x approaches 1?

   b. What seems to be true about the values of y for $g\ (x)$ as the values of x approach 1? Do both representations of the function (table and graph) lead you to the same conclusion?

4. Let $h(x) = \frac{1}{x-1}$.
   a. Complete the table below.

x approaches 1 from the left → ← *x approaches 1 from the right*

x	0.9	0.99	0.999	1	1.001	1.01	1.1
f(x)							

   b. Use a graphing calculator to graph $h(x)$. Use the trace feature of your calculator to examine the values of y as the value of x approaches 1 from both directions. What happens to the values of y as the value of x approaches 1?

   c. What seems to be true about the values of y for $h(x)$ as the values of x approach 1? Do both representations of the function (table and graph) lead you to the same conclusion?

The **limit** of a function $f(x)$ is the one number L that $f(x)$ becomes arbitrarily close to as x approaches, but does not equal, a number c. In other words, when a function has a limit; as x approaches a value c, the y value of the function $f(x)$ approaches the limit value, L.

The notation $\lim_{x \to c} f(x)$ is read as "the limit of $f(x)$ as x approaches c."

5. Use the functions in items 2, 3, and 4 to evaluate the following, if they exist.

    a. $\lim_{x \to 1}(x^2 + 2) =$

    b. $\lim_{x \to 1} \frac{x^2-1}{x-1} =$

    c. $\lim_{x \to 1} \frac{1}{x-1} =$

III. Finding the Limit of a Function

The limit of a function at a particular value of x can be found by using one of several methods. Tables and graphs can each be used to determine the value L that $f(x)$ approaches as x gets close to a particular value, c.

6. Use a table to determine each of the following limits.

    a. $\lim_{x \to 3} \frac{x^3-7x^2+17x-15}{x-3} =$

x	**2.9**	**2.99**	**2.999**	**3**	**3.001**	**3.01**	**3.1**
f(x)							

    b. $\lim_{x \to 0} \frac{\sin x}{x} =$

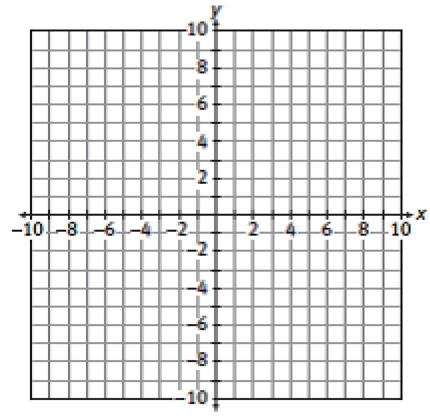
x				**0**			
f(x)							

    c. $\lim_{x \to 1} \frac{x-2}{x^2-4} =$

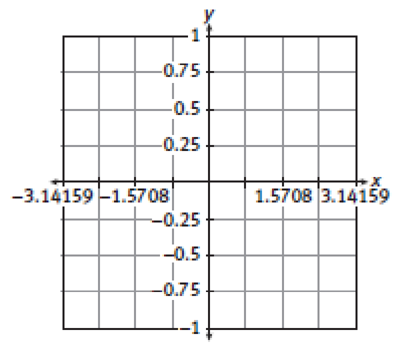
x							
f(x)							

7. Use your graphing calculator to graph each of the following and determine the limit, if it exists. Include a sketch of the graph of the function.

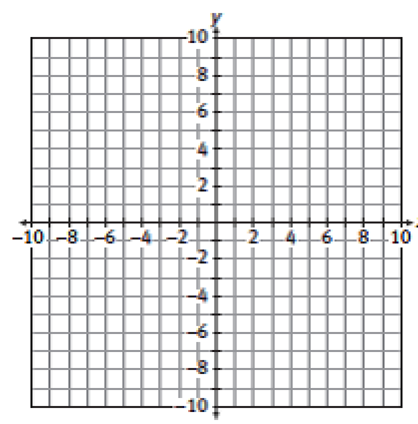
   a. $\lim_{x \to 2}(x^2 + 1) =$



   b. $\lim_{x \to 1} \sin(\pi x) =$



   c. $\lim_{x \to 1} \frac{2x-4}{2x^2-2x-4} =$

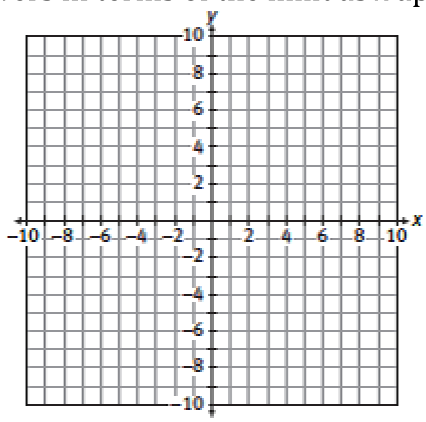


8. For each of the following functions how does the function value at $x = c$, if it exists, relate to the limit of the function as x approaches c ? Use either a graph or a table to find the limit.
   a. $f(x) = x^2 + 2x - 3; \quad x = 3$
   b. $g(x) = -2x^3 + 4; \quad x = 2$
   c. $h(x) = \frac{2}{x^2}; \quad x = 0$
   d. $k(x) = \frac{x+1}{x-4}; \quad x = 4$

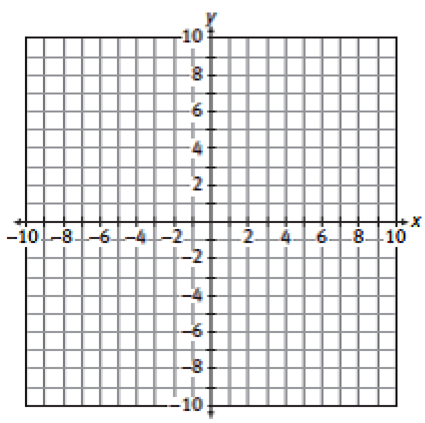
While the limit of a function as x approaches c does not depend on the value of f at $x = c$, it is possible for the limit, L, to equal $f(c)$. When this occurs, the limit can be determined using *direct substitution* so that $\lim_{x \to c} f(x) = f(c)$.

Before using direct substitution to determine limits, it is important to understand some basic limit properties.

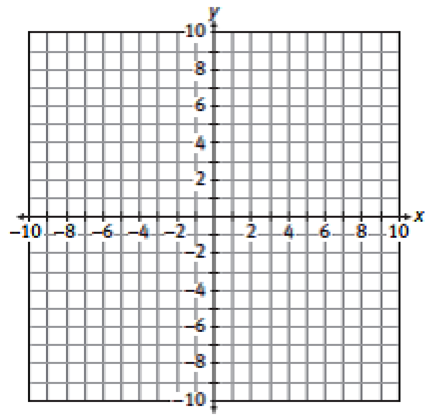
9. Sketch a graph of $f(x) = 5$. Use the graph to find $\lim_{x \to 3} f(x)$, $\lim_{x \to 4} f(x)$ and $\lim_{x \to 1} f(x)$. Explain your answers in terms of the limit as x approaches c of $f(x)$.



10. Sketch a graph of $f(x) = x$. Use the graph to explain why $\lim_{x \to 3} f(x) = 3$.



11. Sketch a graph of $f(x) = x^3$. Use the graph to explain why $\lim_{x \to 2} f(x) = 8$.



12. Use your answers to items 8 through 10 to complete the basic limits below in general form.

Basic Limits

Assume b and c are real numbers and n is a positive integer.

a. $\lim_{x \to c} b =$

b. $\lim_{x \to c} x =$

c. $\lim_{x \to c} x^n =$

13. Use the basic limits to evaluate the following limits by direct substitution.

a. $\lim_{x \to 3} 4 =$

b. $\lim_{x \to 2} x =$

c. $\lim_{x \to 5} x^3 =$

In addition to the basic limits, there are some properties used to evaluate limits by direct substitution.

14. Assume b and c are real numbers and n is any positive integer. Let *f (x)* and g (*x*) be functions such that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$

    Use the information above to complete the following.

    a. *Constant Multiple Property:* Write the informal statement "The limit of a function, f (x), multiplied by a constant b is the constant b multiplied by the limit of the function" in formal limit notation.

    b. *Sum or Difference Property:* Write the informal statement "The limit of the sum or difference of two functions, *f (x)* and *g (x)*, is equal to the sum or difference of the limits of the functions" in formal limit notation.

    c. *Product Property:* Write the informal statement "The limit of the product of two functions is equal to the product of the limits of the two functions" in formal limit notation.

    d. *Quotient Property:* Write an informal verbal statement for the formal limit notation: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}$, where $K \neq 0$.

    e. *Power Property:* Write an informal verbal statement for the formal limit notation: $\lim_{x \to c} [g(x)]^n = K^n$.

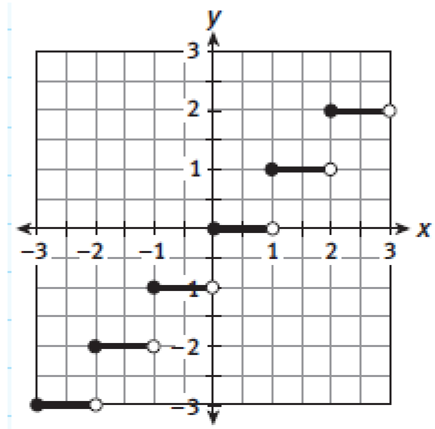
15. Evaluate each of the following limits. Verify each using the limit properties.

    a. $\lim_{x \to 2} x^4 =$

    b. $\lim_{x \to 2} 12x =$

    c. $\lim_{x \to 1} \frac{2x+4}{x-5} =$

d. $\lim_{x \to 4}(3x^2 + 2x - 4) =$



IV. One-Sided Limits

From your exploration of limits, you know that it is possible for $\lim_{x \to c} f(x)$ not to exist. If the graph of $f(x)$ approaches a different value when x approaches c from the left than when x approaches c from the right, the limit does not exist. In this case, it is often helpful to examine a **one-sided limit**. There are two types of one-sided limits. The **limit from the right** describes the limit value as x approaches c from x-values greater than c and uses the notation $\lim_{x \to c^+} f(x)$. The **limit from the left** describes the limit value as x approaches c from x-values less than c and uses the notation $\lim_{x \to c^-} f(x)$. Much like the limit, L, of a function, one-sided limits can be found using tables, graphs, or direct substitution.

16. Use a table to find the limits for the function $f(x)\frac{|3x|}{x}$.

x							
f(x)							

a. $\lim_{x \to 0^+} f(x) =$
b. $\lim_{x \to 0^-} f(x) =$
c. $\lim_{x \to 0} f(x) =$

<u>Note</u>: When a limit does not exist, it is generally accepted to write DNE to mean "does not exist."

17. Use a graph to find the limits for the function $f(x) = \begin{cases} x - 1; x \leq 2 \\ \frac{1}{2}x; x > 2 \end{cases}$.

Include a sketch of the graph.

a. $\lim_{x \to 2^+} f(x) =$
b. $\lim_{x \to 2^-} f(x) =$
c. $\lim_{x \to 2} f(x) =$

18. Use direct substitution to find the limits for the function $f(x) - 2x^3 - 3x^2 + 1$. Show all work.
    a. $\lim_{x \to 1^+} f(x) =$
    b. $\lim_{x \to 1^-} f(x) =$
    c. $\lim_{x \to 1} f(x) =$

19. Using your responses to items 16, 17, and 18, describe the relationship that seems to exist between the values of the one-sided limits of a function and the value of the limit, L, of the function. Give specific examples to support your reasoning.

20. Using direct substitution, $\lim_{x \to 4} \frac{x^2-2x-8}{x-4} = \frac{0}{0}$. Use the methods below to evaluate $\lim_{x \to 4} \frac{x^2-2x-8}{x-4}$. Show all work.
    a. graphing
    b. Table
    c. What do your responses to parts (a) and (b) indicate about the use of direct substitution when finding limits?

21. By factoring, $\lim_{x \to 4} \frac{x^2-2x-8}{x-4} = \lim_{x \to 4} \frac{(x-4)(x+2)}{x-4}$
    a. Is $\lim_{x \to 4} \frac{x^2-2x-8}{x-4} = \lim_{x \to 4}(x+2)$? Explain your reasoning.
    b. Evaluate $\lim_{x \to 4}(x+2)$ by direct substitution.

22. What do your responses to Items 20 and 21 indicate about finding the value of the limit of a function that, when evaluated by direct substitution, yields the indeterminate form $\frac{0}{0}$?

23. Assume the following limits have been evaluated by direct substitution to yield the indeterminate form given. Reevaluate each to find the value of the limit, if it exists. Explain your answers.
    a. $\lim_{x \to 0} \frac{x}{x^2} = \frac{0}{0}$
    b. $\lim_{x \to 1} \frac{x^2-1}{x+1} = \frac{0}{0}$
    c. $\lim_{x \to 2} \frac{x-2}{(x-2)^3} = \frac{0}{0}$

24. Assume the following limits have been evaluated to yield the expressions given. What conclusions can be made about each limit? Explain your reasoning for each.

a. $\lim_{x \to 4} f(x) = \frac{3}{3}$

b. $\lim_{x \to 4} f(x) = \frac{0}{2}$

c. $\lim_{x \to 4} f(x) = \frac{1}{0}$

d. $\lim_{x \to 4} f(x) = \frac{0}{0}$