8.2: Number Theory

Extension Activity

Number theory is a branch of mathematics devoted primarily to the study of integers. Number theorists study prime numbers as well as the properties of objects made out of



integers (e.g., rational numbers) or defined as generalizations of the integers (e.g., algebraic integers).

Pictured above is a Lehmer sieve, which is a primitive digital computer once used for finding primes and solving simple Diophantine equations.

1. Some discoveries in number theory involve numbers that are impossibly large such as Skewes' numbers and Graham's number.

One Skewes' number is approximately $e^{(e^{(e^{79})})}$ and Graham's number is so large that to even write it requires 64 lines of writing with a new operation (one that can be thought of as the shortcut for repeated exponentiation).

In fact, both of these numbers are so large that the decimal representation of the numbers would be larger than the known universe and dwarf popular large numbers such as googol and googolplex (10^{100} and $10^{(10^{100})}$ respectively). These large numbers, although nearly impossible to comprehend, are still not at the "end" of the real numbers, which have no end. Consider the function $f(x) = x^2 - 10^{100}$.

a. Consider only positive values of *x*; how long until f(x) > 0?

b. If your answer to part (a) represented seconds, how many billions of years would it take for f(x) > 0? (Note: one billion years is approximately 3.15×10^{16} seconds). How close is this to the estimated geological age of the earth (4.54 billion years)?

- c. Number theorists frequently only concern themselves with the term of a function that has the most influence as $x \to \infty$. Let $f(x) = x^3 + 10x^2 + 100x + 1000$, and answer the following questions.
 - x^3 100*x* 1000 $10x^{2}$ $10x^{2}$ x^3 100*x* х f(x)f(x) $\overline{f(x)}$ f(x)f(x)0 10 100 1000
 - i. Fill out the following table:

ii. As $x \to \infty$, which term of f(x) dominates the value of the function?

iii. Find
$$g(x) = \frac{f(x)}{x}$$
. Which term dominates $g(x)$ as $x \to \infty$?

d. Consider the formula for a general polynomial, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ for real numbers $a_i, 0 \le i \le n$. Which single term dominates the value of f(x) as $x \to \infty$.