7.9: Diving Deeper into Function Composition

Composing Functions

Freediving, also known as apnea diving is both a recreational and a competitive sport, in which divers descend into the water and then swim to the surface… all without any breathing apparatus!

Freediving can be dangerous to competitors because the rapid change in pressure the divers experience as they descend can cause nitrogen bubbles to form in their capillaries, inhibiting blood flow. Some freedivers also experience shallow water blackouts and lose consciousness during their ascent.

For more information about freediving, view the New Yotrker article "The Deepest Dive" at http://www.newyorker.com/magazine/2009/08/24/the-deepest-dive

Consider the tables below:

Depth of Free Diver During Descent									
s seconds of descent	20	40	60	80	100	120	140	160	180
d depth in meters of diver	15	32	44	65	79	90	106	120	133

Atmospheric Pressure and Ocean Depth									
d depth in meters of diver	10	20	30	40	50	60	70	80	90
p pressure in atm on diver	2	3	4	5	6	7	8	9	10

1. Do the tables appear to represent functions? If so, define the function represented in each table using a verbal description.

2. What are the domain and range of the functions?

3. Let's define the function in the first table as $d = f(s)$ and the function in the second table as $p = g(d)$. Use function notation to represent each output, and use the appropriate table to find its value:

   a. depth of the diver at 80 seconds

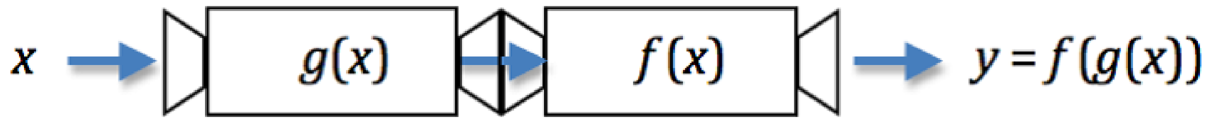
   b. pressure of the diver at a depth of 60 meters

   c. Explain how we could determine the pressure applied to a diver after 120 seconds of descending.

   d. Use function notation to represent part (c), and use the tables to evaluate the function.

   e. Describe the output from part (d) in context.

4. Explain how the following diagram might help you think about the work you have been doing on the previous problems. How does the notation used in the diagram support the way you have been combining functions in this task? This way of combining functions is called **function composition**.



Imagine you are working at a dive school that provides lessons and sells equipment. The software the company uses keeps track of student info including the dates of all the classes taken and who the instructors were.

Consider the following report functions:

- f: Name→ Course Date
  This function assigns to each person the date he or she passed the dive course
- g: Name→ Name
  This function assigns to each person the name of his or her instructor

5. Describe the action of each composite function <u>and</u> state whether the composite function makes sense in the real-world context.

   a. $g(f)$

   b. $g(g)$

   c. $f(g)$

   d. $f(g(g))$

II. Composition of Functions

Let's formalize our definition of a composite function.

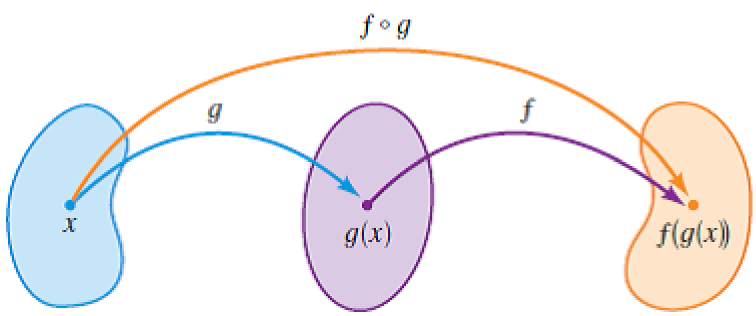
COMPOSITION OF FUNCTIONS

Given two functions f and g, the **composite function $f \circ g$** (also called the **composition** of f and g) is defined by

$$(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))$$

Note: We read both $f(g)$ and $f \circ g$ as "f of g."

Note 2: Pay attention to the order in which you apply the functions, as indicated in the arrows below:



EXAMPLE 1 – Evaluating a Composition

Let $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}^2$ and $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x} - \mathbf{3}$. Find $(\boldsymbol{f} \circ \boldsymbol{g})(\mathbf{5})$ and $(\boldsymbol{g} \circ \boldsymbol{f})(\mathbf{7})$

SOLUTION:

- $(\boldsymbol{f} \circ \boldsymbol{g})(\mathbf{5}) = \boldsymbol{f}\big(\boldsymbol{g}(\mathbf{5})\big) = \boldsymbol{f}(\mathbf{2}) = \mathbf{2}^2 = \mathbf{4}$
- $(g \circ f)(7) = g\big(f(7)\big) = g(49) = 49 - 3 = 6$

6. Use $f(x) = 3x - 5$ and $g(x) = 2 - x^2$ to evaluate the expression.
   a. $(f \circ g)(3)$

   b. $(g \circ f)(3)$

   c. $(f \circ g)(-1)$

   d. $(f \circ f \circ g)(0)$

EXAMPLE 2 – Finding the Composition of Functions

Let $\boldsymbol{f(x) = \sqrt{x}}$ and $\boldsymbol{g(x) = \sqrt{2 - x}}$. Find the following compositions <u>and</u> determine the domain and range of the composite function:

a. $(\boldsymbol{f} \circ \boldsymbol{g})$
b. $(\boldsymbol{g} \circ \boldsymbol{f})$

SOLUTION:

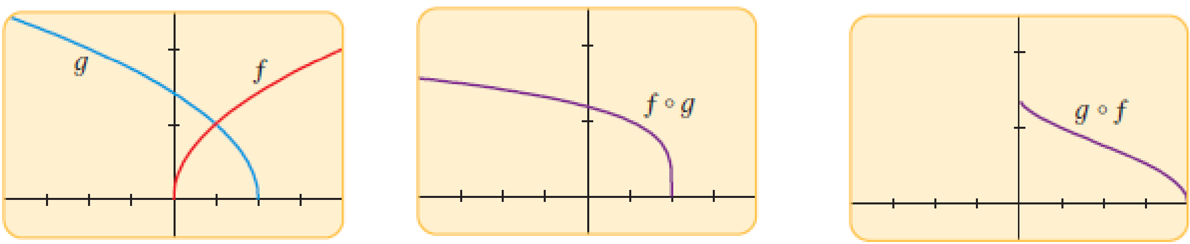
a. $$\begin{aligned}(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x})) &= \boldsymbol{f}(\sqrt{\boldsymbol{2 - x}}) \\ &= \sqrt{\sqrt{\boldsymbol{2 - x}}} \\ &= \sqrt[4]{\boldsymbol{2 - x}}\end{aligned}$$

- The domain of $(\boldsymbol{f} \circ \boldsymbol{g})$ is $\boldsymbol{x \leq 2}$ (since anything less than 2 would result in a negative number inside the radical.)
- The range of $(\boldsymbol{f} \circ \boldsymbol{g})$ is all positive real number.

b. $$\begin{aligned}(g \circ f)(x) = g(f(x)) &= g(\sqrt{x}) \\ &= \sqrt{2 - \sqrt{\boldsymbol{x}}}\end{aligned}$$

- Domain: For $\sqrt{x}$ to be defined, we must have $x \geq 0$. For $\sqrt{2-\sqrt{\boldsymbol{x}}}$ to be defined, we must have $2-\sqrt{x} \geq 0$, that is $\sqrt{x} \leq 2$, or $x \leq 4$. Therefore the domain of $(g \circ f)$ is $0 \leq x \leq 4$.
- The range of $(g \circ f)$ is all positive real number.

The graphs below demonstrate the domains and ranges of the compositions. Also note, how different are the shapes of the composed functions from the original functions.



<u>Your Turn</u>:

7. Let $\boldsymbol{f}(\boldsymbol{x}) = \sqrt{\boldsymbol{x}-\boldsymbol{3}}$ and $\boldsymbol{g}(\boldsymbol{x}) = \sqrt{\boldsymbol{3}-\boldsymbol{x}}$. Find the following compositions <u>and</u> determine the domain and range of the composite function:

   a. $(\boldsymbol{f} \circ \boldsymbol{g})$

   b. $(\boldsymbol{g} \circ \boldsymbol{f})$

8. Suppose a sports medicine specialist is investigating the atmospheric pressure placed on competitive free divers during their descent. The following table shows the depth, d, in meters of a free diver s seconds into his descent. The depth of the diver is a function of the number of seconds the free diver has descended, $d = f(s)$.

s seconds	10	35	55	70	95	115	138	160	175
d depth in meters	8.1	28	45	55	76.0	91.5	110	130	145

The pressure p, in atmospheres, felt on a free diver, is a function of his or her depth, $p = g(d)$.

d meters	25	35	55	75	95	115	135	155	175
p atm	2.4	3.5	5.5	7.6	9.6	11.5	13.7	15.5	17.6

a. How can the researcher use function composition to examine the relationship between the time a diver spends descending and the pressure he or she experiences? Use function notation to explain your response.

b. Explain the meaning of $g(f(0))$ in context.

c. Use the charts to approximate these values, if possible. Explain your answers in context.

i. $g(f(70))$

ii. $g(f(160))$

III. Applications

9. According to the Global Wind Energy Council, a wind turbine can generate about 16,400 kilowatt hours (kWh) of power each day. According to the Alternative Fuels Data Center, an average electric car can travel approximately 100 miles on 34 kWh of energy. An environmental nonprofit organization is interested in analyzing how wind power could offset the energy use of electric vehicles.

   Write a function that represents the relationship between the number of wind turbines operating in a wind farm and the amount of energy they generate per day (in kWh). Define the input and output.

   Write a function that represents the relationship between the energy expended by an electric car (in kWh) and the number of miles drive

   Write a function that could be used to determine the number of miles that an electric car could drive based on the number of wind turbines operating daily at a wind farm. Interpret this function in context.

   Determine an appropriate domain and range for part (c). Explain why your domain and range are reasonable in this context.

   How many miles of driving could be generated daily by 20 wind turbines in a day?

10. A department store manager is planning to move some cement spheres that have served as traffic barriers for the front of her store. She is trying to determine the relationship between the mass of the spheres and their diameter in meters. She knows that the density of the cement is approximately 2500 kg/m^3.
    a. Write a function that represents the relationship between the volume of a sphere and its diameter. Explain how you determined the equation.

    b. Write a function that represents the relationship between the mass and the volume of the sphere. Explain how you determined the function.

    c. Write a function that could be used to determine the mass of one of the cement spheres based on its diameter. Interpret the equation in context.

    d. Determine an appropriate domain and range for part (c).

    e. What is the approximate mass of a sphere with a diameter of 0.9 meter?