Primes and Composites

> The positive integers stand there, a continual and inevitable challenge to the curiosity of every healthy mind.
>
> **G.H. Hardy** (English mathematician; 1877 - 1947)
>
> It will be another million years, at least, before we understand the primes.
>
> **Paul Erdo¨s** (Hungarian mathematician; 1913 - 1996)
>
> Were it not for your [Duke of Brunswick] unceasing benefits in support of my studies, I would not have been able to devote myself totally to my passionate love, the study of mathematics. **C.F. Gauss** (German mathematician; 1777 - 1855)

Recall that prime numbers are those positive integers whose only divisors are 1 and the number itself. Any positive integer that is not prime is called **composite**. Any number that evenly divides another positive integer is called a **factor** of the latter.

For example, the number 462 is composite since it is even. We can *completely factor* 462 into its prime factorization $462 = 2 \times 3 \times 7 \times 11$. 2,3,7, and 11 are all factors of 462, as are combinations of them like 6,14, and 77. The *prime factorization* of 462 is called complete because none of the factors can be broken down any further - all factors are prime. If we start with a composite number, we are able to factor it, then factor the factors, and then factor these smaller factors, and continue until all of the factors are primes.

Not only can every positive integer be completely factored into primes, but the representation is unique up to the order in which the factors appear. This result is called the **fundamental theorem of arithmetic**. It is truly of fundamental importance for it says is that the prime numbers are, via multiplication, the building blocks of the positive integers. As elements serve as the building blocks for all chemical compounds, the primes serve as the building blocks for the positive integers.

Contemporary mathematicians do not consider the number 1 to be prime.

2	3	5	7	11	13	17	19
23	29	31	37	41	43	47	53
59	61	67	71	73	79	83	89
97	101	103	107	109	113	127	131
137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223
227	229	233	239	241	251	257	263
269	271	277	281	283	293	307	311
313	317	331	337	347	349	353	359
367	373	379	383	389	397		

The primes under 400.

462 has precisely four prime factors - 2,3,7,11. If 1 were considered a prime number then it would be a factor as well. And in fact, it could be a repeated factor: $462 = 1 \times 1 \times 1 \times 2 \times 3 \times 7 \times 11$. This opens a flood-gate of complications. It is this then that lead to the decision that we should not consider 1 to be a prime number.

Twin Primes and Other Arithmetic Progressions of Primes

The numbers 5 and 7 are called **twin primes** because they come in pairs, as close as two odd primes can be. 11 and 13, 17 and 19, and 29 and 31 are other twin prime pairs. Mathematicians have long known there are infinitely many primes, they have been unable to determine whether there are infinitely many twin prime pairs or not. Most believe there infinitely many - this belief is called the **twin prime conjecture** - but this remains a major open question in number theory. Solve this mystery and you will be a mathematical celebrity.

1. Find all of the twin primes under 100.

2. Find all of the twin primes between 100 and 200.

3. Find all of the twin primes between 200 and 300.

4. Find all of the twin primes between 300 and 400.

5. What do you notice about the number of twin primes in each of these ranges?

6. Why do you think this might be happening?

7. Find a *progression* of three twin primes; that is, a sequence of three primes each that is a twin prime to the one that follows it.

8. Prove that no other progression of three twin primes can exist.

9. Can you find a progression of twin primes longer than three terms? If not, why? If so, how large of a progression can you find?

Cousin primes are a pair of prime numbers that are four away from each other. **Sexy primes** are a pair of primes that are six away from each other.

10. Find several pairs of cousin primes.

11. Can you find a progression of three cousin primes? If one progression exists, are there others? (Prove your answer.)

12. Can you find a progression of cousin primes that is longer than three terms? If not, why? If so, how long of a progression can you find?

13. Find several pairs of sexy primes.

14. Can you find a progression of three sexy primes? If one progression exists, are there others? (Prove your answer.)

15. Can you find a progression of sexy primes that is longer than three terms? If not, why? If so, how long of a progression can you find?

16. Make up your own name for a pair primes that are eight apart. Explain your name.

17. Find several pairs of eight-apart primes.

18. Can you find a progression of three eight-apart primes? If one progression exists, are their others? (Prove your result.)

19. Can you find a progression of eight-apart primes that is longer than three terms? If not, why? If so, how long of a progression can you find?

20. How long of a progression of 30-apart primes can you find?

In 1910, **Edward B. Escott** (; -) discovered that all of the terms in the progression

199, 409, 619, ..., 1669

were prime.

21. How far apart are the numbers on Escott's list?

22. How many numbers are on Escott's list?

23. Looking back at your examples, which "apart numbers" generate fairly long progressions of primes?

24. What properties do these useful "apart numbers" share? How are these useful "apart numbers" related to each other?

From 1910 - 1963, mathematicians were not able to find a longer run of consecutive *primes in arithmetic progression* than Escott. At the time of this writing (2014) the longest run of consecutive primes in arithmetic progression that is known is 26.[1] On the basis of this evidence, it is stunning that one can *prove* that for *any* finite length there *must* exist a run of consecutive primes in arithmetic progression of at least this length. In other words, despite being stuck at 26, we know that there is an arithmetic progression of one billion primes in a row! This startling result was proven in 2004 by **Ben Green** (British mathematician; 1977 -) and **Terence Tao** (Australian mathematician; 1975 -).[2] At age 31, Tao won mathematics' highest honor - the Fields medal - in part for his work on this problem.

The Green/Tao theorem, as it is called, is an *existence theorem* which proves existence but does not tell you how to actually construct the desired objects. It gives us no clue how to find these elusive arithmetic progressions of primes that can exceed any finite length. The Green/Tao theorem had many additional consequences, many outlined in a paper by a premier number theorist named **Andrew Granville** (British mathematician; 1962 -).[3] Here you will investigate one such prime pattern.

The array of numbers

3	7
19	23

is called a 2 × 2 *generalized arithmetic progression of primes* (GAP) because all of the numbers are prime and along each row the numbers are 4 apart and along each column the numbers are 16 apart.

25. Find another 2 × 2 generalized arithmetic progression of primes.

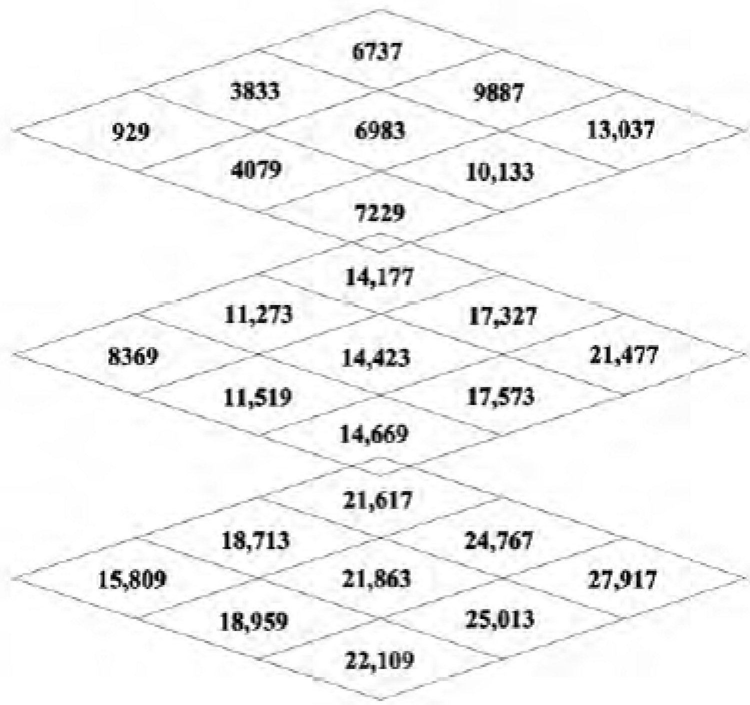
26. There is 3×3 generalized arithmetic progression of primes whose smallest member is 29 and whose largest member is 113. Find the other members of this GAP.

The Green/Tao theorem insures that larger and larger such GAPs exist, including a Rubik's cube like 3×3×3 GAP. One such GAP is illustrated on the next page.

[1] Search "Primes in arithmetic progression records" to learn if this record has been extended since this writing.

[2] Green, Ben and Tao, Terence (2008), "The primes contain arbitrarily long arithmetic progressions", *Annals of Mathematics* 167 (2): 481?547.

[3] "Prime Number Patterns", *The American Mathematical Monthly*, vol. 115, No. 4, April 2008, pgs. 279-296.



3 × 3 × 3 GAP by Guenette, Vanasse, Fleron and Jaiclin.