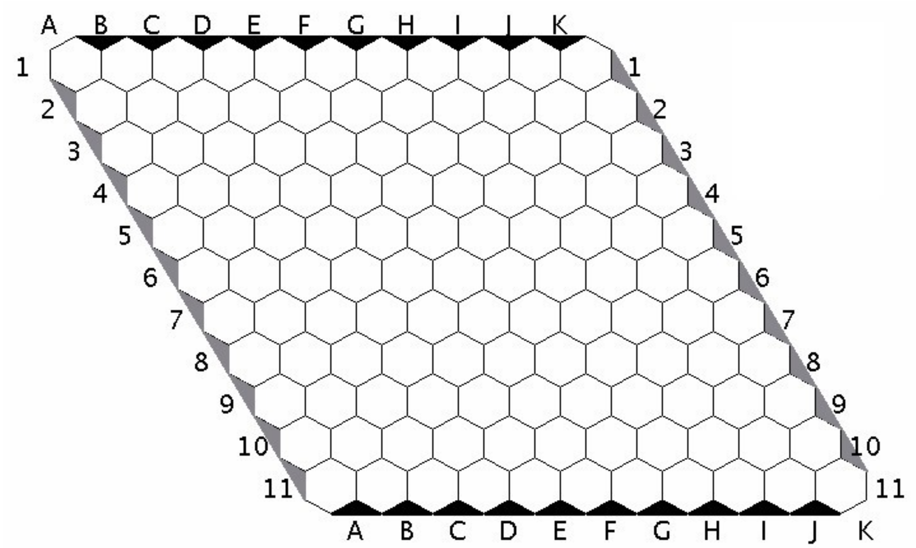


Game Trees

We are interested in how many different Hex games we can play on Hex boards of various sizes. As you may have seen, some games end fairly quickly, while others require us to fill up much of the board, so this is a difficult question to get a good handle on. Often in mathematics, when faced with a question that is too difficult, it can be helpful to consider an easier question first.

So let's think about a simpler question first: Imagine we have a new game called "Fill-it-Up" which is just like Hex, except that the players need to continue until the entire board is filled up. Ignore for the moment that the game is not really interesting to play once one person has made a connection.

We consider two "Fill-it-Up" games to be the same when every single move by the two players is exactly the same, in exactly the same order. So two games are different if somewhere along the line, one of the players makes a different move (think of writing down a complete record of every single move: *A*1-*B*3-*...*; when two of those lists agree, the game is the same, otherwise, it differs). Now the question I'd like you to consider is: How many different "Fill-it-Up" games are there?

Let's consider for example a 2 × 2 board. Black goes first and puts a stone in *A*1. Next, gray places a stone in *A*2. Followed by black in *B*2 and gray in *B*1. Now the board is completely filled so we stop. A different example: *B*2-*A*2-*A*1-*B*1 (black starts and player take turns). Notice that once all the moves are done, the boards look identical, but we got there in different ways, so **38.** In how many ways can you fill up a 2 × 2 board (taking turns between Gray and Black)?

1. How many possibilities do you have to place the first stone?

2. Given that there is now one stone on the board, how many possibilities do you have to place a second stone?

3. Given that there are two stones on the board, how many possibilities do you have to place the third stone?

4. What pattern do you observe? Explain.

5. Now consider a 3×3 board? What would the pattern look like here? In how many ways can you fill up this board?

6. How about a 4 × 4 board?

7. Describe a process to find out in how many ways we could fill up a 14 × 14 board.

8. How does the number of different "Fill-it-Up" games relate to the total number of possible **Hex** games? (Which is what we are really interested in.)

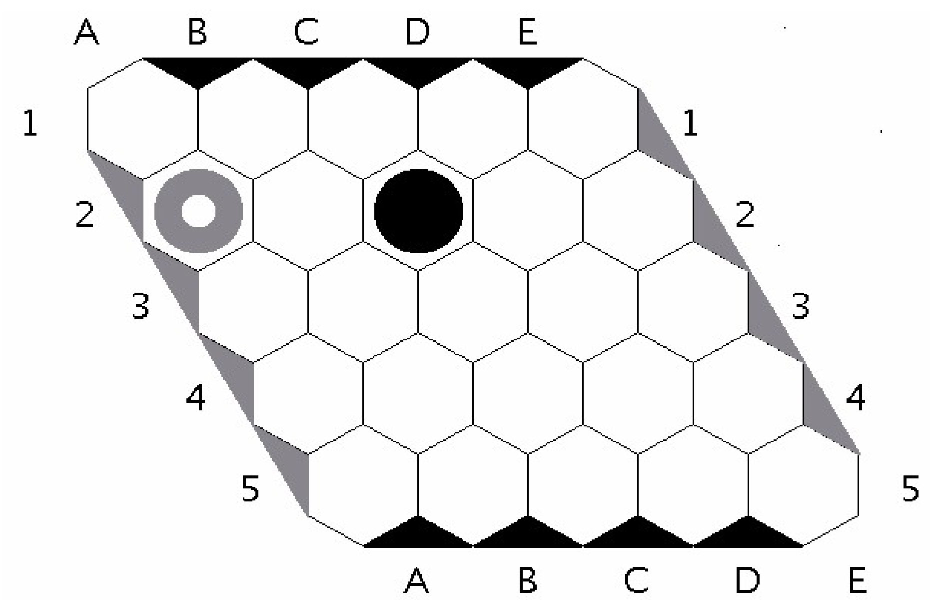
Size of board *n*	Number of hexagons on *n* × *n* board	Number of ways to fill up the board
n = 2 *n* = 3 *n* = 4 *n* = 14		

You may need a calculator, or even a computer with Maple or Mathematica to work out these numbers in detail.

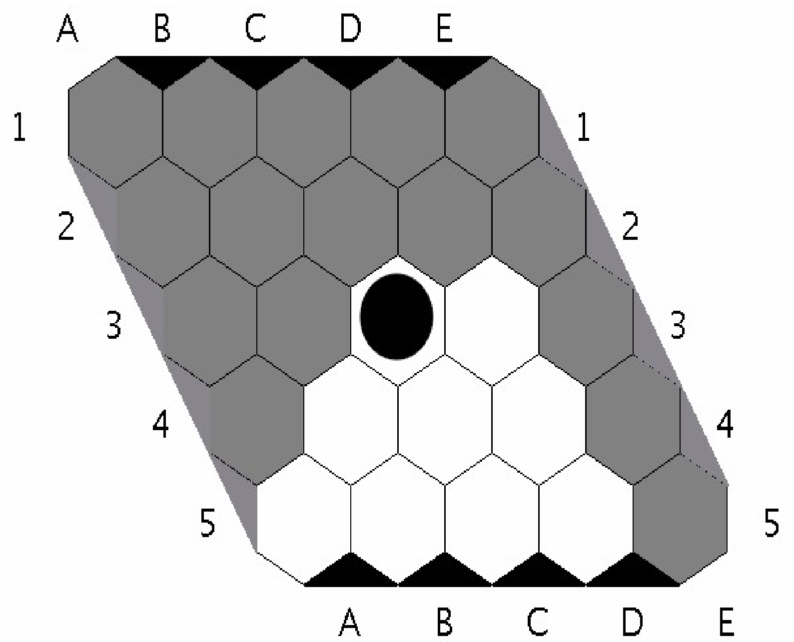
This method over-counts the total number of possibilities. Cameron Browne shows a more careful accounting, with the following numbers.

Size of board *n*	Valid Game Positions
n = 1	2
n = 2	17
n = 3	2844
n = 4	4,835,833

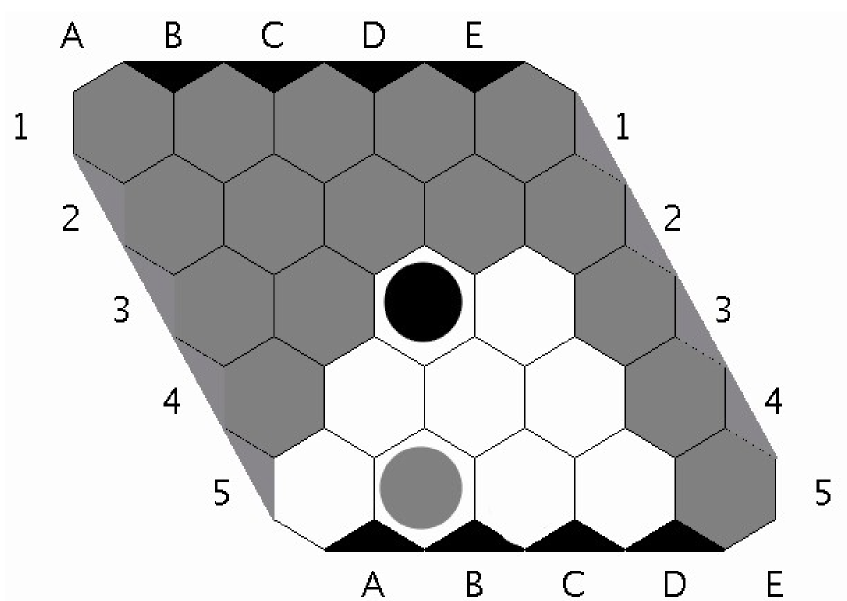
9. How could you program a computer to win in every game?



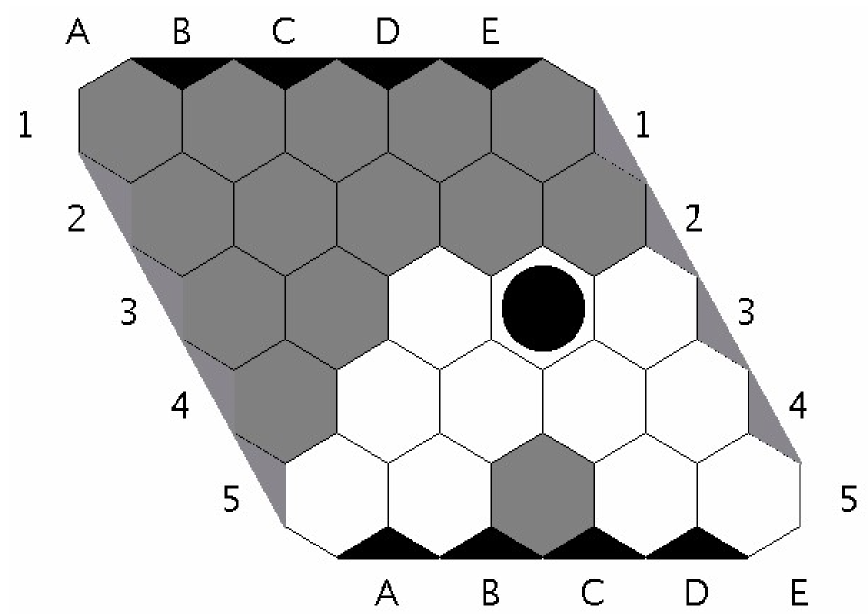
10. In the figure above, it is Black’s turn to play. Which is the best move?



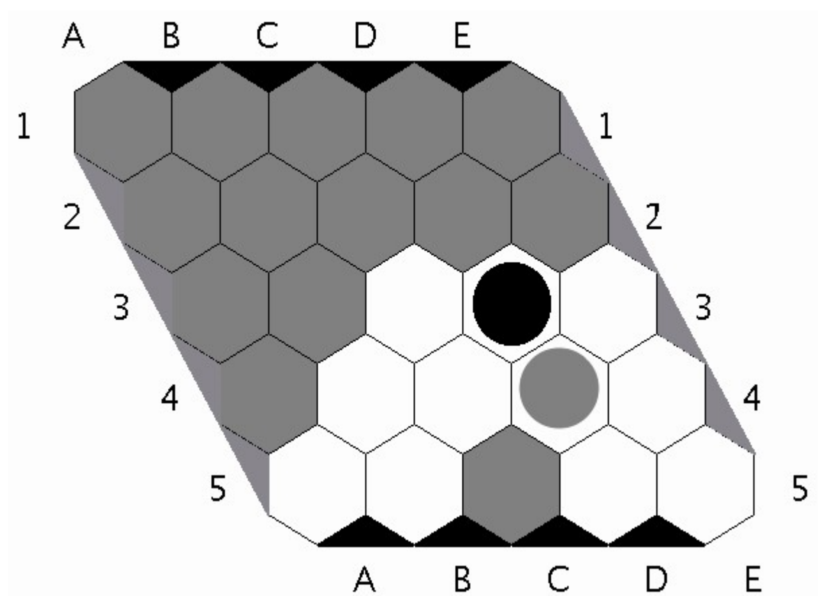
11. In the figure above, how can Gray block Black from reaching the edge?



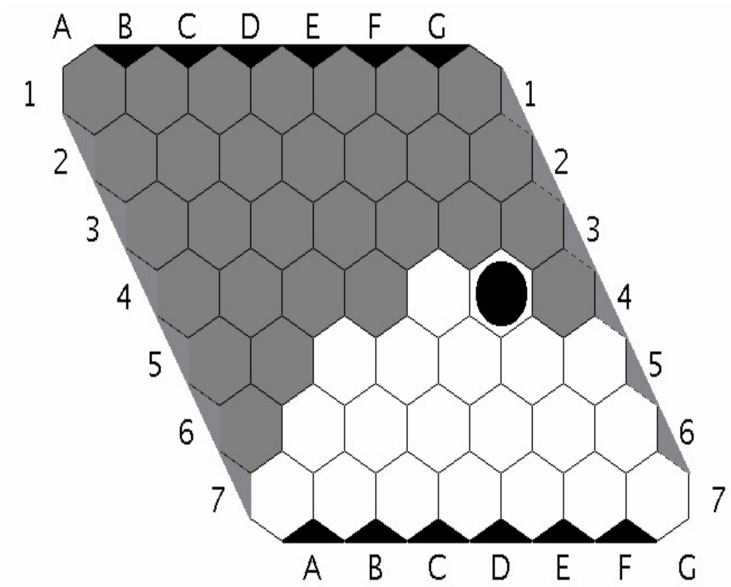
12. In the figure above and playing for Black, what would your strategy suggest?



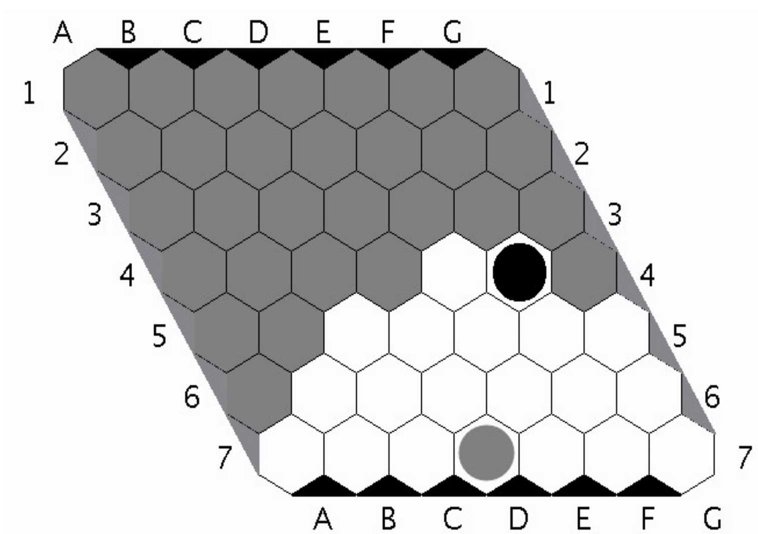
13. In the figure above, how can Gray block Black from reaching the edge?



14. In the figure above, playing for Black, what would your strategy suggest?



15. In the figure above, how can Gray block Black from reaching the edge?



16. Playing for Black, what would your strategy suggest?