Pythagorean Rationalism

All is number.

Pythagoras (Greek mathematician and philosopher; circa 570 BC - circa 495 BC)

Pythagoreans celebrate sunrise, painting by **Fyodor Bronnikov**(1827–1902)

The Pythagoreans were a cult-like group of followers of Pythagoras who had a significant influence on Greek intellectual culture for several centuries, having a significant influence on greats like Plato and Aristotle. The Pythagoreans were mystics, philosophers, early scientists and musicians. Fundamental to their beliefs was the essential role of number. In addition to their number mysticism, they used number to explain musical scales, to seek to explain the motions of heavenly bodies, and to understand geometry.

By “number” the Pythagoreans meant whole numbers and fractions constructed from whole number ratios. Such numbers are now called rational numbers.

1. Do you believe that every number is rational? I.e. can every number be represented as a fraction? Explain carefully.

2. Show that the numbers 7, -6, and 11 can be represented as fractions.

3. Show that the numbers 1 and -1 can be represented as fractions.

4. Correct to three decimal places the number $\sqrt{2} \approx 1.414$. Can you write this latter number as a fraction? Do so or explain why it cannot be done.

5. Correct to ten decimal places the number $\sqrt{2} \approx 1.4142135623$. Can you write this latter number as a fraction? Do so or explain why it cannot be done.

6. Correct to twenty decimal places the number $\sqrt{2} \approx 1.41421356237309504880$. Can you write this latter number as a fraction? Do so or explain why it cannot be done.

7. Correct to thirty decimal places the number $\sqrt{2} \approx 1.414213562373095048801688724209$. Can you write this latter number as a fraction? Do so or explain why it cannot be done.

The process in Investigation 5 - Investigation 7 can be continued indefinitely, getting arbitrarily close to the exact value of $\sqrt{2}$. In fact, the process can be repeated for every number which is represented by a finite or even infinite decimal. This property of the rational numbers is called *density*.

Let us restrict our attention to the rational numbers - they will be our microworld. Microworlds call for action and exploration, so what can we do?

8. Take several pairs of fractions and multiply each pair together. Is the product of each pair a fraction?

9. Does your work in Investigation 8 prove anything about products of fractions?

10. We can write a general pair of fractions as $\frac{a}{b}$ and $\frac{c}{d}$. What can you say about the numbers *a,b,c,d*? I.e. what type of numbers are they and are their any limitations on their values?

11. Can you write the product $\frac{a}{b} \cdot \frac{c}{d}$ as a fraction? If so, do so. If not, explain why not.

12. What does Investigation 11 prove? Explain.

13. Take several pairs of fractions and add each pair together. Is the sum of each pair a fraction?

14. Does your work in Investigation 13 prove anything about sums of fractions?

15. Can you write the sum $\frac{a}{b} + \frac{c}{d}$ as a fraction? If so, do so. If not, explain why not.

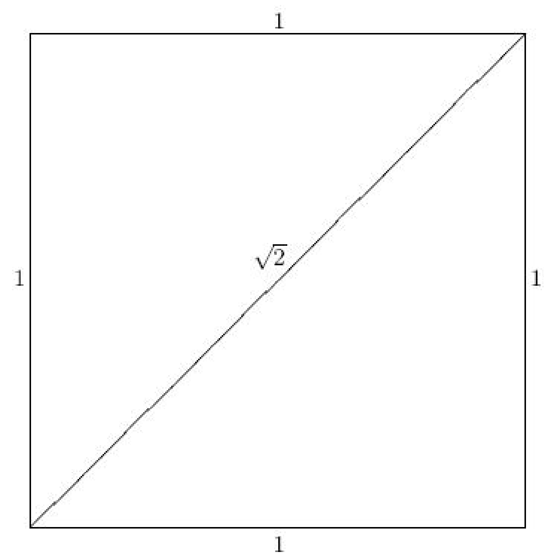
16. What does Investigation 15 prove? Explain.

We can continue in this way to show that all of the standard operations of arithmetic - addition, subtraction, multiplication and division - apply for fractions. We can also find additive and multiplicative inverses. As such, the rational numbers form what is known as a *field*.

For all of these reasons, a fundamental tenet of Pythagorean mathematics was the following conjecture:

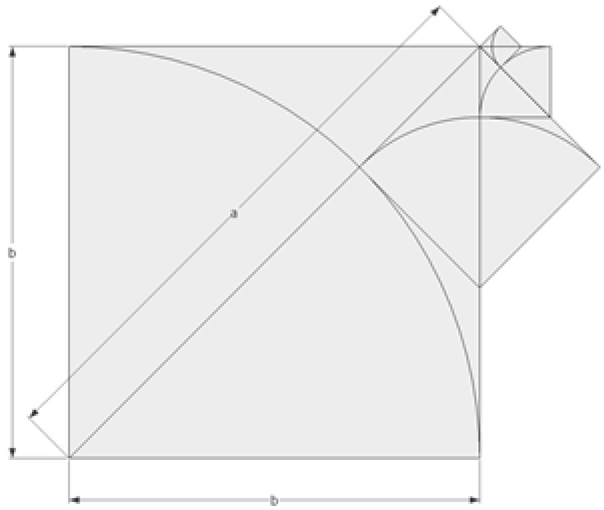
> Conjecture (The Rational Number Microworld of the Pythagoreans): Every number can be represented as a fraction. There are no other numbers.

It turns out this conjecture is false. There are irrational numbers; numbers that are not commensurable with a unit length. So damning and heretical was this discovery that legend has it that the person who shared this discovery outside of the circle of the Pythagoreans, **Hippasus of Metaponum** (Greek philosopher; circa 5th century B.C.), was drowned as punishment.



The first irrational number to be found was nothing esoteric or mysterious, but the commonplace $\sqrt{2}$ which is the length of the diagonal of the unit square. Some 2,000 years later mathematicians would discover that in a probabilistic sense the rational numbers are a vanishingly small proportion
of the real numbers.

There are many proofs that $\sqrt{2}$ cannot be represented as a fraction. Euclid gave an algebraic proof. There are many different geometric proofs as well. The images below is the basis of one of the geometric proofs.



17. Find one of these proofs, understand how it works, and then rephrase it in your own words, as you would explain it to a non-mathematical audience.