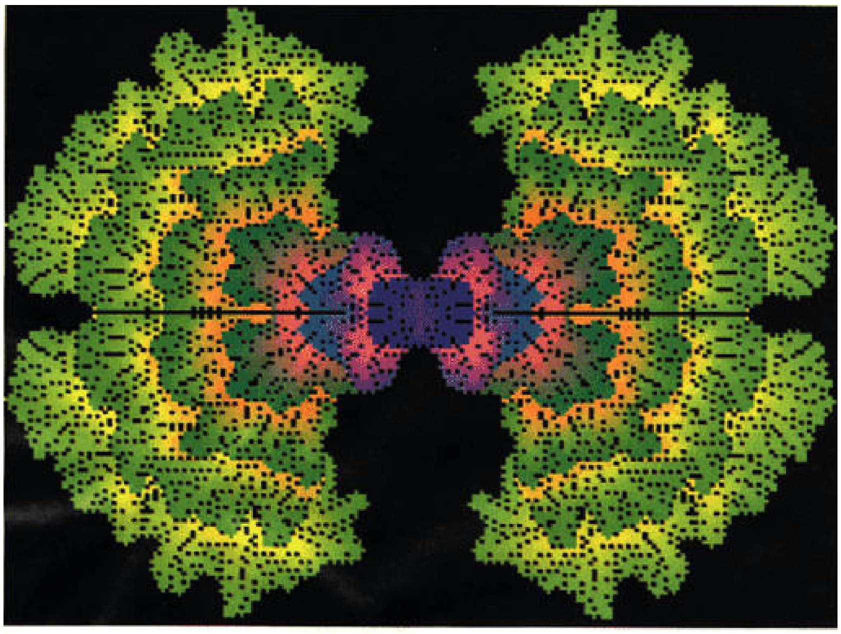
A New Kind of Science



How Everything Works

Stephen Wolfram says science has been broken for more than 300 years—and he can fix it

In 2002 **Stephen Wolfram** (British Mathematician and Computer Scientist; 1959 -) self-published the magnum opus *A New Kind of Science*. The importance of this highly anticipated book continues to be vigorously debated. Wolfram is not shy of touting the impact of "his" work:

> Three centuries ago science was transformed by the dramatic new idea that rules based on mathematical equations could be used to describe the natural world. My purpose in this book is to initiate another such transformation, and to introduce a new kind of science that is based on much more general types of rules that can be embodied in simple computer programs.[1]

This "New Science", Wolfram tells us, is:

[1] *A New Kind of Science*, p. 1

> ...one of the more important single discoveries in the whole history of theoretical science. For in addition to opening up vast new domains of exploration, it implies radical rethinking of how processes in nature and elsewhere work.[2]

If one didn't know better the term "crackpot" might seem appropriate. But Wolfram's accomplishments are remarkable. He earned a Ph.D. in physics from the California Institute of Technology at age 20. At age 21 he earned a prestigious MacArthur Fellowship (a.k.a. Genius Award). At 24 he was on the faculty at the Institute for Advanced Study at Princeton University. At 28 he cofounded Wolfram Research which is responsible for the design and development of the profoundly important mathematical software Mathematica and more recently the computational knowledge engine Wolfram Alpha. This company has made him a millionaire.

Wolfram describes one of his fundamental conclusions in A New Kind of Science as follows:

> One might have thought - as at first I certainly did - that if the rules for a [computer] program were simple then this would mean that its behavior must also be correspondingly simple. For our everyday experience in building things tend to give us the intuition that creating complexity is somehow difficult, and requires rules or plans that are themselves complex. But the pivotal discovery that I made some eighteen years ago is that in the world of programs such intuition is not even close to correct.[3]

The importance of this observation?

[In human building and engineering] we restrict ourselves to systems whose behavior we can readily understand and predict - for unless we can foresee how a system will behave, we cannot be sure that the system will do what we want. But unlike engineering, nature operates under no such constraint. So there is nothing to stop systems like those at the end of the previous sections from showing up. And in fact one of the important conclusions of this book is that such systems are very common in nature.[4]

You will now investigate two simple systems that are similar to those that underlie Wolfram's "New Science."

Iteration and the Collatz Conjecture

1. Pick a positive whole number. Then apply the rule below to your number. What is the result of your evaluation?

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[2] *A New Kind of Science*, p. 2
[3] *A New Kind of Science*, p. 2.
[4] *A New Kind of Science*, p. 40.

Collatz Rule: If the number is even, divide it by two. If the number is odd, multiply by three and then add one.

2. Now apply the Collatz Rule to the result of Investigation 1. What is the result of your evaluation?

3. Now apply the Collatz Rule to the result of Investigation 2. What is the result of your evaluation?

4. Apply the Collatz Rule at least a dozen more times, each time using the result of the previous application as the next number for which the rule is applied. Describe the results of your evaluations. (One typical way of writing the iterates is to denote the seed value by N_0, the first iterate by N_1, etc.)

5. Pick another starting value, often called a seed, and repeat Investigations 1 - 4 for this seed value.

6. Repeat Investigation 5 for a third seed.

7. Repeat Investigation 5 for a fourth seed.

The process of repeatedly applying a rule or function to the previous output is called **iteration** of the rule/function. Starting with a specific seed value the sequence of outputs is called the orbit of the rule/function for this seed value.

In 1937 **Lothar Collatz** (German Mathematician; 1910 - 1990) stated what has become known as the Collatz conjecture:

Collatz Conjecture. Choose any positive integer seed value. If you iterate the Collatz rule repeatedly, eventually the output will reach 1 and then continue indefinitely as 1,4,2,1,....

8. Compare your results with your peers. Do all of your results agree with Collatz's conjecture? Explain.

The Collatz conjecture remains unsolved to this day and the great character **Paul Erdos** (Hungarian Mathematician; 1913 - 1996) even says of it, “Mathematics is not yet ready for such confusing, troubling, and hard problems.”

The Collatz rule can be expressed algebraically as a simple mathematical function:

$$c(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$

Let's see what happens when we iterate a simpler function. Let's consider the function

$$s(n) = n + 1.$$

9. Iterate the function s starting from the seed value $n = 1$.

The Number of Prime Numbers

Here you will investigate another iterative process, one which provides a beautiful proof of the infinitude of the prime numbers. The rule for this iteration, which we will call the **Saidak rule**, is as follows. At each stage the result is the product of two numbers:

- The previous result, and,
- The number one larger than the previous result.

Example: Suppose our seed value is $N_0=2$. Then our iterates are:

$$\begin{aligned} N_1 &= 2 \times 3 = 6 \\ N_2 &= 6 \times 7 = 42 \\ N_3 &= 42 \times 43 = 1806 \\ N_4 &= 1806 \times 1807 = 3263442 \\ &\vdots \end{aligned}$$

10. Find the first five iterates when the seed value is $N_0 = 3$.

11. Choose your own starting seed value N_0. Find the first five iterates corresponding to this seed value.

12. Repeat Investigation 11 for a new seed value of your choice.

13. Repeat Investigation 11 for one more seed value of your choice.

We would like to determine which prime numbers evenly divide each of the iterates. Such divisors are naturally called the **prime factors**. As shown in the example below, we can assemble all of the prime factors into the *prime factorization* of each iterate:

$$
\begin{aligned}
N_0 &= 2 = 2\\
N_1 &= 6 = 2 \cdot 3\\
N_2 &= 42 = 2 \cdot 3 \cdot 7\\
N_3 &= 1806 = 2 \cdot 3 \cdot 7 \cdot 43\\
&\vdots
\end{aligned}
$$

14. Notice that in each of the first three iterations, the prime factorization of the iterate includes exactly one new prime factor in addition to prime factors of the iterate that came before it. If this continued to happen indefinitely, what would this tell us about the number of prime numbers that exist?

15. Find the prime factorization of the iterate $N_4 = 3263442$. Does it include exactly one more prime factor?

16. Find the prime factorizations of the iterates from Investigation 11 . What do you notice about the prime factors and the number of prime factors that are created as we iterate?

17. Find the prime factorizations of the iterates from Investigation 12. What do you notice about the prime factors and the number of prime factors that are created as we iterate?

18. Find the prime factorizations of the iterates from Investigation 13. What do you notice about the prime factors and the number of prime factors that are created as we iterate?

It should be clear that the common factors - prime or non-prime - of a number, n, and its immediate successor $n + 1$ are important in this iterative process.

19. Choose a positive integer n. What common factors do n and $n + 1$ share?

20. Repeat Investigation 18 for several other integers n.

21. State a conjecture about the common factors that any positive integer n and its immediate successor $n + 1$ share.

22. Prove the conjecture in Investigation 21.

23. Having established this conjecture deductively, this should enable you to definitively establish a result about number of prime factors that result from each iteration of the Saidak rule.

24. Explain how this proves that there are infinitely many prime numbers. Does the seed value, N_0, of the iteration matter?

This is an amazing thing - you have deductively established that there are infinitely many prime numbers.

We've suggested investigations to approach this task. Return for a moment and analyze which steps are absolutely necessary. Which of the investigations were there to help with the learning but are not required as part of a *rigorous proof*?

25. Write a complete, rigorous, entirely self-contained proof that there are infinitely many primes as economically as you can.

Earlier you rediscovered Euclid's proof of the infinitude of the primes from *The Elements*. Many, many other proofs exist. The simple and elegant proof rediscovered above is only 5 years old.[5] It was discovered (or should we say invented?) by **Filip Saidak**. He thought of this proof while waiting for a bus at a bus stop!

> Proofs aren't there to convince you that something is true - they're there to show you why it is true.
>
> **Andrew Gleason** (American mathematician; 1921 - 2008)

[5] "A new proof of Euclid's theorem," *American Mathematical Monthly*, vol. 113, no. 10, December 2006, pp.