Tessellations

The figure at the left shows a tiled floor. Because the floor is entirely covered by the tiles we call this arrangement a **tessellation** of the plane.

A **regular tessellation** occurs when:

- The tessellation must tile a plane surface (that goes on forever) with no overlapping or gaps.
- The tiles must be regular polygons - and all of them *congruent*.
- Each vertex must look the same.

1. Draw an isosceles triangle. Show how you can tessellate the plane with your triangle. Seeing this might make you wonder if you can tessellate with an arbitrary triangle.

2. Draw a non-isosceles triangle. Now make a template of this triangle out of cardstock or cardboard.

3. On a large sheet of paper or chalkboard, use the template to draw a copy of your triangle. Now see what ways you can find to put another copy of your triangle together with the first to begin to create a tessellation.

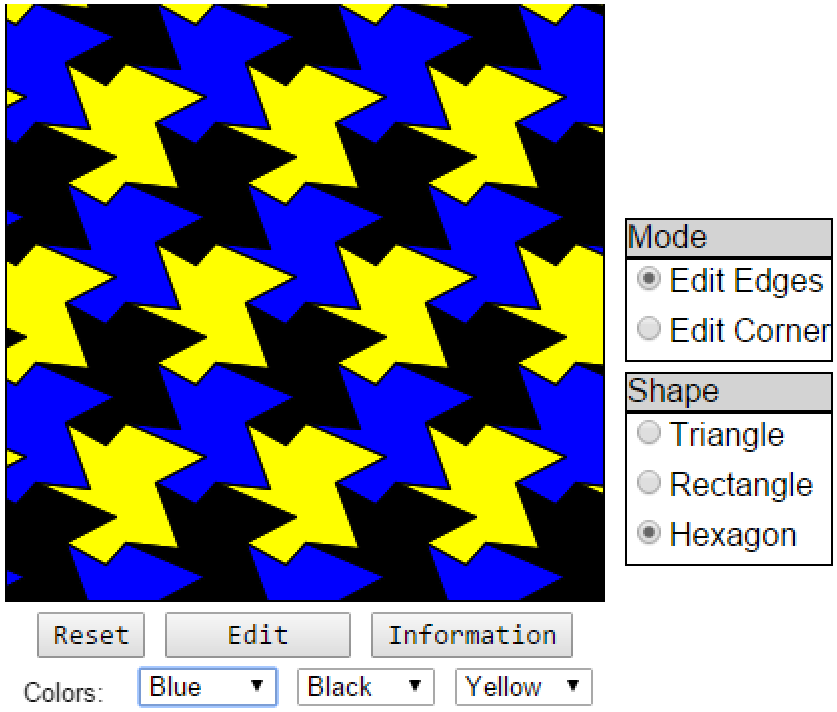
4. Now add several more triangle copies as you try to tessellate. You may have to try several different arrangements - perhaps even very early in your arrangement.

5. Eventually you should be able to find a way to tessellate, regardless of which triangle you started with. Do so. Describe how the orientation of adjacent triangles in your tessellation are related. How might your tessellation be colored to illustrate this pattern?

6. Look at any vertex of your tessellation. Which angles surround this vertex? Is this true at every vertex?

7. Use this to prove the Triangle Sum Theorem for your triangle.

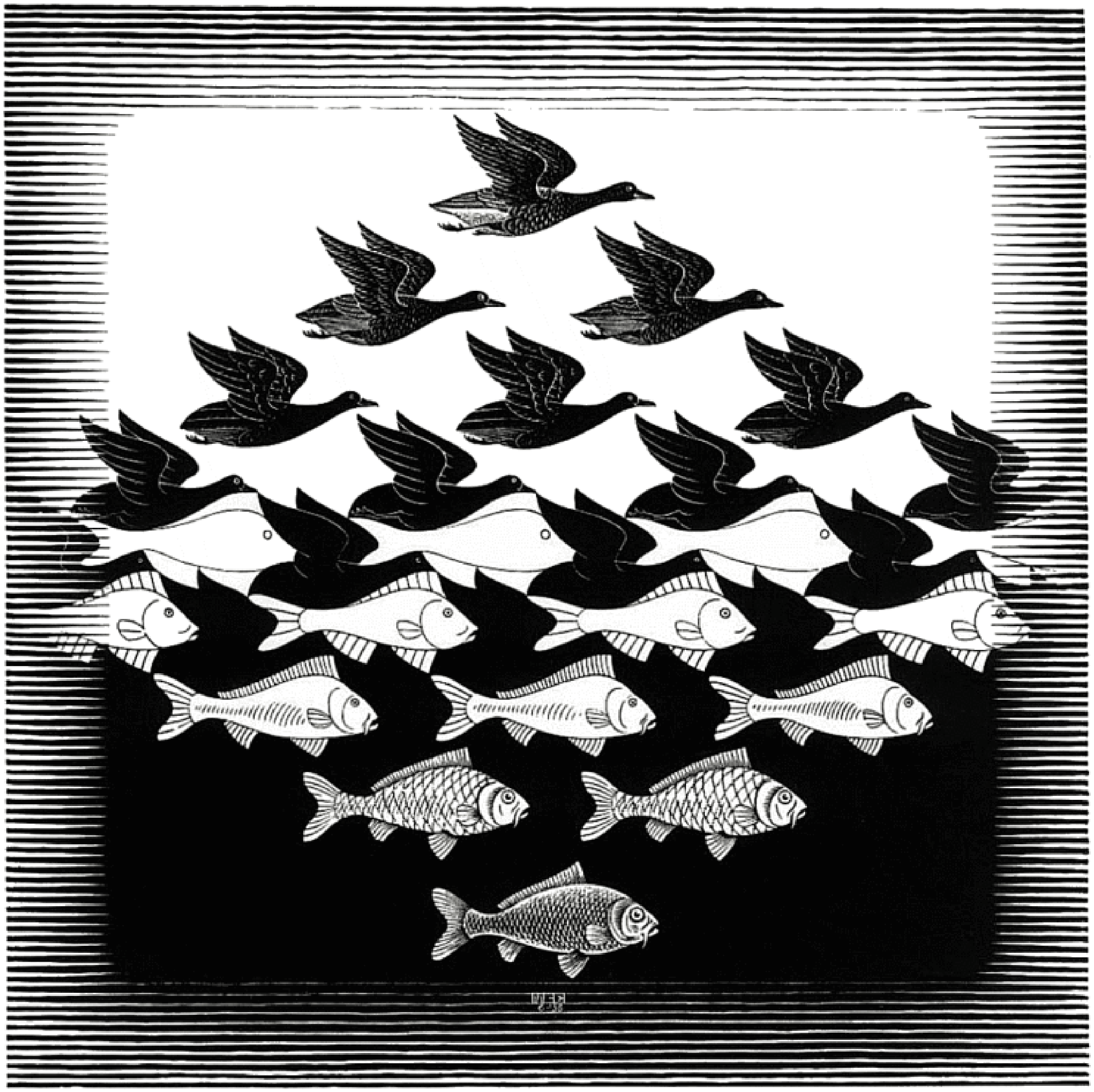
8. The applet available at http://www.shodor.org/interactivate/activities/Tessellate/ allows you to dynamically change the shape of the initial triangle and see what happens to your tessellation. Do these tessellations provide a proof of the Triangle Sum Theorem? Why or why not?



9. Create three original irregular tessellations based on a) a triangle, b) a rectangle and c) a hexagon. (Hint "Edit Edges" for each side of the polygon.)

10. Why do these irregular shapes tessellate?

11. Describe the tessellation present in the following graphic design.

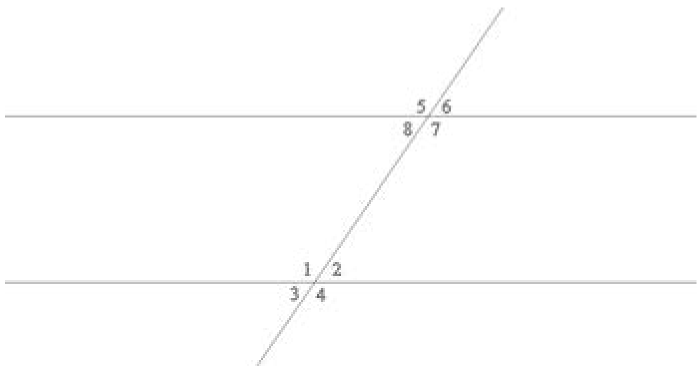


Sky and Water I, M. C. Escher 1939 woodcut print

Parallel lines cut by a different transversal

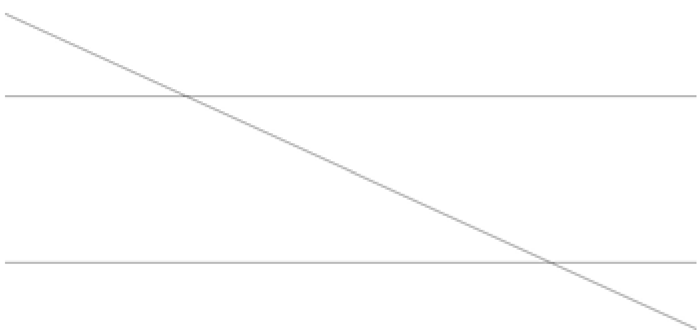
One of the earliest proofs of the Triangle Sum Theorem is reminiscent of your construction above. It appears as Proposition 32 in Book I of *Euclid's Elements*.

Euclid's proof relies on a fundamental result about parallel lines that he proves as Proposition 29 in Book I.



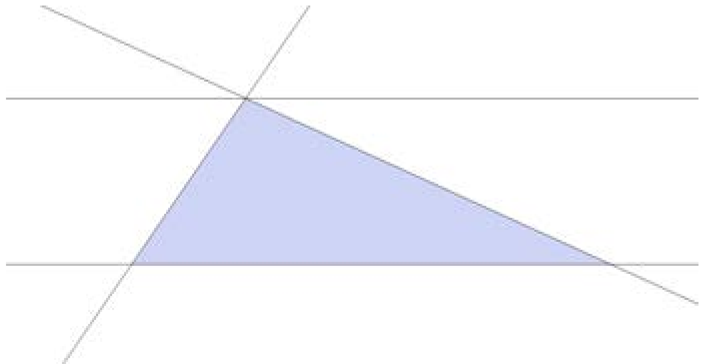
12. Pictured above is a pair of parallel lines cut by another line, usually called a transversal. How are the angles labeled 1 - 8 related to one another?

13. Pictured below is the same set of parallel lines cut by a different transversal. How are the angles in this figure related to one another?



14. Is there a general result that holds for every transversal that intersects these two lines? If so, state it as a theorem and explain - at least intuitively - how you know this result is true. If not, illustrate the limitation with an example.

15. The figure below shows both transversals cutting the pair of parallel lines. Use your results above to prove the Triangle Sum Theorem for this one triangle.
If you are given a different triangle can this process be repeated? Explain. Does this give you a proof of the general result? Explain.



The triangle formed by the two different transversals

Triangle Sum Theorem Conclusions

16. Which of the investigations above help you most in understanding why the Triangle Sum Theorem is true?

17. Which proof of the Triangle Sum Theorem is most compelling to you?

A Famous Proof

In Lesson 4.2 - Proofs Without Words, you investigated the following sums:

Determine the value of the following sums:

- 1 + 3 =
- 1 + 3 + 5 =
- 1 + 3 + 5 + 7 =
- 1 + 3 + 5 + 7 + 9 =

It is hoped that you made the following conjecture:

<u>Conjecture</u>: If the number $2 \cdot 3 \cdot 5 \ldots p_n + 1$ is formed where $2, 3, 5, \ldots, p_n$ are consecutive primes, then any prime divisor of this number must be greater than p_n.

18. If you form the number $2 \cdot 3 \cdot 5 \ldots p_n + 1$ is formed where $2, 3, 5, \ldots, p_n$ are consecutive primes, can any of the primes $2, 3, 5, \ldots, p_n$ divide this number? Explain.

19. Explain how the previous Investigation proves the Conjecture above.

20. Explain how this establishes, deductively, the following theorem:

Theorem. Let the number $N_n = 2 \cdot 3 \cdot 5 ... p_n + 1$ be formed where $2,3,5,...,p_n$. Then either:

- N_n is prime, or,
- N_n is divisible by some prime larger than p_n.

21. Suppose that there were only finitely many primes. Then there would have to be a largest prime. Denote this largest prime by $p_{Largest}$. If we apply the previous theorem, what happens? What does this tell you about our assumption.

22. Explain how our results prove:
Theorem (Euclid). There are infinitely many primes.

23. Does it seems strange to you that you were able to prove that there were infinitely many primes without providing a way of explicitly generating prime numbers indefinitely? Explain.

The theorem above was proven by Euclid in his famous collection *The Elements*. The proof here is the original Euclidean proof. It is among the most famous proofs in all of mathematics.