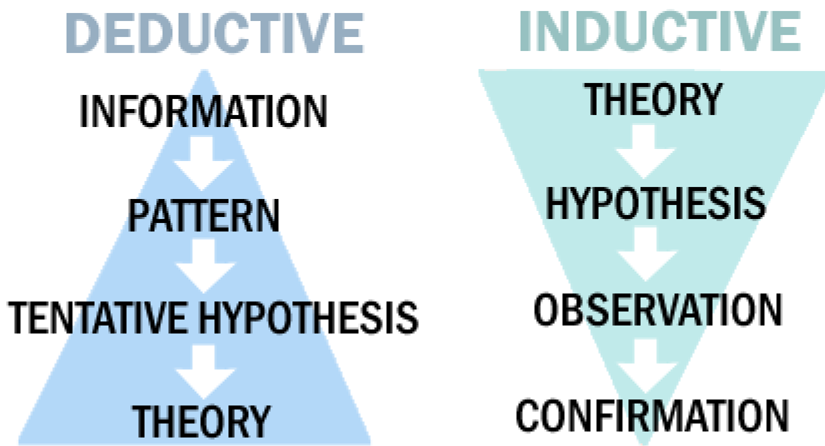

Types of Reasoning

> Science is the attempt to make the chaotic diversity of our sense-experiments correspond to a logically uniform system of thought.
>
> **Albert Einstein** (German physicist and author; 1879 - 1955)

Here we will investigate two critical types of reasoning - inductive and deductive. We shall see that each deals with patterns, relationships, connections, causes, prediction, control, explanation, and implications, but in very different ways.

What's My World is a game in which one person, who we'll call the creator, creates an imaginary world from a set of rules. The creator provides clues about their imagined world while the other players attempt to guess the rules that define this imaginary world. Each clue is given in the form "In my world there are ______ but no ______."

To help find the rules, players can ask the Creator whether certain things are in the world and the clue returned by the Creator will be given in the same form. Sample clues for a game are:

"In my world there are birds but no cats."
"In my world there are gnats but no children."
"Is there chocolate in your world?" "In my world there are geese but no chocolate."

1. Guess what the rules are that defines this world. Are you confident?

2. In fact, these clues have been set up so there are several different possible answers. Find a few more.

3. What are some questions you could ask to help distinguish which of your guesses may be correct and which are not?

4. Other than asking directly, is there any way you can be absolutely certain that you have discovered the defining rule?

5. Choose a Creator and play What's My World. Describe the clues, how the game progressed and whether the rules were guessed.

6. Is there any way, other than asking the Creator, that you could be certain that you had determined the rules of this world? Explain.

> This universe, I conceive, like to a great game being played out, and we poor mortals are allowed to take a hand. By great good fortune the wiser among us have made out some few of the rules of the game, as at present played. We call them "Laws of Nature," and honour them because we find that if we obey them we win something for our pains. The cards are our theories and hypotheses, the tricks our experimental verifications. But what sane man would endeavor to solve this problem? The problem of the metaphysicians is to my mind no saner.[1]
>
> **Thomas Henry Huxley** (English biologist; 1825 - 1895)

7. Use the quote by Thomas Huxley to explain how What's My World is related to the scientific method. Give several examples from your own world where this process of trying to find rules has lead you to false conclusions.

8. Consider the sequence William Henry Harrison, Abraham Lincoln, James A. Garfield, and William McKinley. Why are these four people on these list together? Can you think of a reason that these particular people were chosen? Explain.[2]

9. Suppose we add Warren G. Harding to the end of this list. Does your answer to the previous question still hold? If not, can you posit a new principle which governs our list?

10. Franklin D. Roosevelt would be next on the list. And then John F. Kennedy. Can you now guess what the list is?

11. There seems to be a pattern. Some principle in action. So perhaps it would be appropriate to make predictions based on this pattern? What predictions might be made? Did they come to fruition?

---

[1] Quoted on p. 513 of *The Colossal Book of Mathematics* by Martin Gardner.

[2] This example is from Instructor's Guide to Mathematics: A Human Endeavor, 3rd edition, by Harold R. Jacobs.

$$
\begin{aligned}
6 &= 3+3 \\
8 &= 3+5 \\
10 &= 5+5 \\
12 &= 5+7 \\
\vdots & \quad \vdots
\end{aligned}
$$

12. The vertical **ellipsis** $\vdots$ appear below the four equations above because they suggest that the equations form a pattern that continues. What do you think the next five equations are in this pattern?

13. Your answer to Investigation 12 may “fit” the data, but the typical answer does not “match” the pattern we were looking for. Here’s a clue - the number 9 is never allowed. Can you find a different pattern that “fits” this new clue about the pattern?

14. These investigations are a bit like What’s My World. Here are some more clues about the sequence of equations we have in mind - in addition to not using 9 anywhere, you cannot use any even number on the right. Nor can you use 15s. Can you find a pattern that “fits” these new clues?

15. The “rule” that governs the sequence of equations that is intended above gives rise to a conjecture. Use your observations above to complete the following:

Conjecture 1. (Goldbach’s) Every even number $\geqslant 6$ can be written as the ________ of ________ ________.

16. See if you can check this conjecture for the first 25 examples.

We call this **Goldbach’s conjecture** because it was first posed by **Christian Goldbach** (German lawyer and mathematician; 1690 - 1764), an otherwise little-remembered mathematician, in a letter to the great **Leonard Euler** (Swiss mathematician; 1707 - 1783) in a letter dated 7 June, 1742.

The first examples above came from a game. The sequence of dead Presidents is pretty strange, but there couldn’t be any underlying cause for this could there? But now with Goldbach’s conjecture perhaps you have some sense that there must be some underlying cause, rationale, reason, effect, or agent at work.

In fact, the scientific method relies very much on the observation of patterns, relationships, connections, causes, prediction, control, explanation, and implications. The scientist hopes to find a theory that “fits” the data.

Inductive reasoning is the process of drawing general conclusions from limited (usually empirical) evidence. If the conclusions "fit" the data well enough they become known as scientific theories.

Challenges to Inductive Reasoning

In the next several investigations we would like to explore how many prime numbers there are.

17. Complete each of the following computations and determine if the resulting number is prime or not:

$2 + 1 =$
$2 \cdot 3 + 1 =$
$2 \cdot 3 \cdot 5 + 1 =$
$2 \cdot 3 \cdot 5 \cdot 7 + 1 =$
$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 + 1 =$

18. Do you think that the pattern in Investigation 17 will continue indefinitely? If so what type of reasoning are you using? If not, why not?

19. If the pattern in Investigation 17 did go on forever, what could you conclude about the number of primes?

20. Complete the calculation

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1,$$

and then determine if the resulting number is prime. If it is not prime, completely factor the number into prime factors.

21. Repeat Investigation 20 for the number $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 + 1$.

22. Repeat Investigation 20 for the number $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 + 1$. (Hint: 347.)

23. What do these last results tell you about inductive reasoning?

24. In the preceding Investigations, if the number formed was not prime then what can you say about its prime factors? Do you think this pattern will continue? Why?

In a recent paper[3], **Lenny Jones** (American mathematician), introduced a very interesting sequence of numbers:

$$s_0 = 12$$

[3] "When does appending the same digit repeatedly on the right side of a positive integer generate a sequence of composite integers?" American Mathematical Monthly, vol. 118, no. 2, February 2011, pp. 153-60

$s_1 = 121$
$s_2 = 1{,}211$
$s_3 = 12{,}111$
$s_4 = 121{,}111$
⋮

25. Find all of the prime factors of each of the numbers s_0,s_1,s_2 and s_3.
s_4 has a *prime factorization* of $s_4 = 281 \times 431$.

26. Determine which, if any, of the numbers s_5,...s_9 are prime. For those that are not, keep track of any factors that you find in the table below. (Finding all prime factors may be difficult without suitable calculator or computer software.)

n	Term s_n	Factors
0	12	
1	121	
2	1211	
3	12111	
4	121111	281, 431
5	1211111	
6	12,111,111	
7	121,111,111	
8	1,211,111,111	
9	12,111,111,111	
10	121,111,111,111	61, 1985428051
11	1,211,111,111,111	
12	12,111,111,111,111	
13	121,111,111,111,111	
14	1,211,111,111,111,111	
15	12,111,111,111,111,111	
16	121,111,111,111,111,111	683, 177322271026517
17	1,211,111,111,111,111,111	

Notice that factors for a few of the terms in the sequence have been filled in.

27. See what factors you can find for the remaining entries in the table above.

A positive integer that is not prime is called **composite**. When we check to see if a number (resp. collection of numbers) is (resp. are) prime we are checking their *primality*.

28. Do you feel comfortable making a conjecture about the primality of all numbers in the sequence $\{s_n\}$? Explain.

29. Prove that all of the numbers $s_1, s_3, s_5, \ldots$ are all composite.

30. Prove that all of the numbers $s_0, s_3, s_6, \ldots$ are all composite.

31. Find another infinite subsequence of numbers from the sequence $\{s_n\}$ that you think will be composite. Give your reasons for making this conjecture.

32. You have now shown that a very large proportion of the numbers in the sequence $\{s_n\}$ infinitely many of them - are composite. Would you like to amend your answer to Investigation 28? Explain.

33. If you still had concern that maybe some terms in the sequence are prime, identify a specific term you would like to know about. Explain why you chose this term.

34. Each of the numbers s_0 - s_{100} are in fact composite. So in addition to your three infinite subsequences that are composite, all of those terms in the first hundred that might be worrisome are in fact composite as well. Would you now like to amend your answer to Investigation 28? Explain.

35. Do you have any information that pertains to s_{136}? Explain.

36. Write out s_{136} in its entirety.

37. How does one prove that a number is prime? How long do you think it would take to check that s_{136} is prime if, in fact, it is? Explain.

38. Can you guess what the punch-line is going to be?[4]

39. What does this suggest about inductive reasoning?

[4] Yes, s_{136} is the first term in the sequence that is prime.