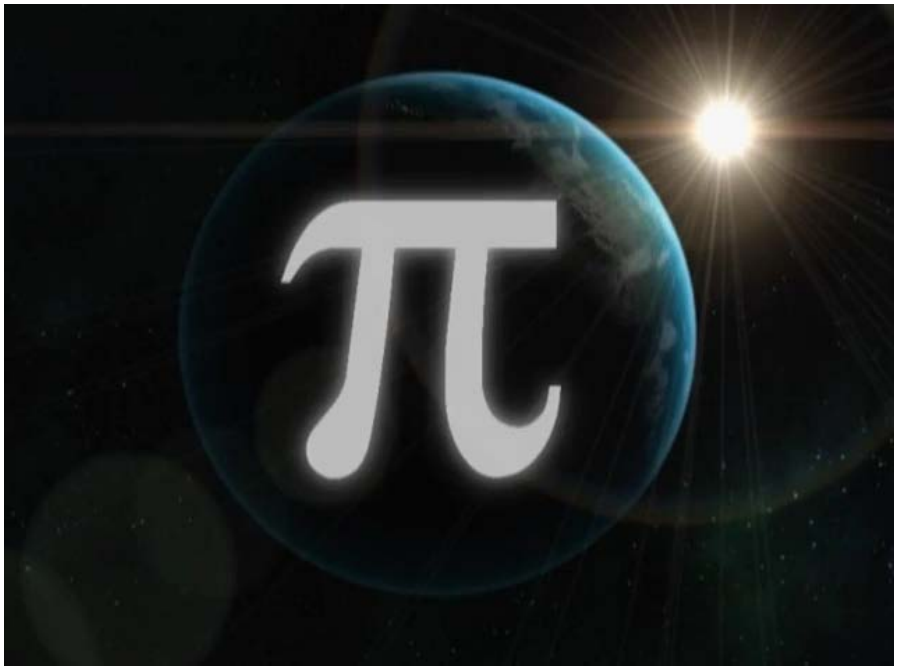


More Mysteries of π

Area πs

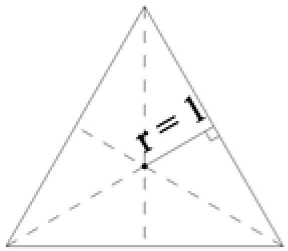
π also arises in the area formula circle. Why is that so? We could send you out to measure the areas of some circles, but measuring the area of circles is quite hard. Archimedes was a major contributor to our understanding of circular areas and perimeters.

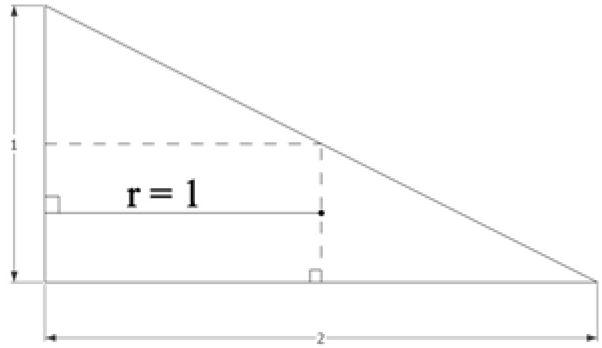
Instead, here we'll return to general geometric shapes.

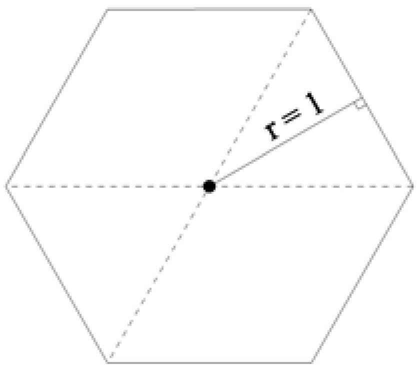
1. Why does the area formula for a circle, $A = \pi r^2$, involve r^2 and not simply r?

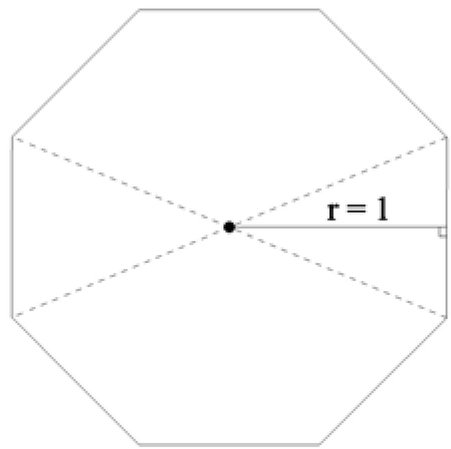
2. Do you think the area formulas for our families of shapes should all involve r^2 even though the perimeters involved only r? Explain.

3. For each of the shapes below determine the area of the shape.









4. Determine the ratio $\frac{Area}{r^2}$ for each shape. What do you notice?

5. Scale the original shape so the radius is scaled from 1 to r where r > 0 is any scaling factor. Determine the area, A, and the ratio $\frac{A}{r^2}$.

6. You now have proven a formula for the area of your chosen shape as it is scaled to any size. Compare and contrast this formula with the formulae $A = \pi r^2$ for circles. Does it make sense to say that you have found a π-like area constant for your chosen shape? Explain.

As with perimeters, this behavior is perfectly natural from the perspective of dimension. The area is a measure of the interior of the shape, the interior being two-dimensional. The length indicated by r is one-dimensional. As the shape is scaled by a factor of m the new indicated length is mr, the area of the new interior is m^2A and the ratio of the new area to the square of the new length is:

$$\frac{m^2A}{(mr)^2} = \frac{A}{r^2}.$$

As with perimeters, this ratio is unchanged!

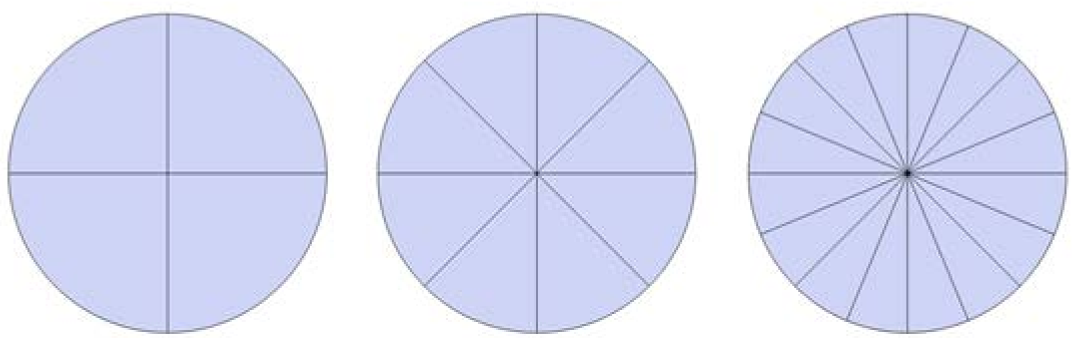
At this stage in the discussion with perimeters we noted that there was no clear evidence in the historic record that the ancient Greeks were able to prove that this ratio is constant. In contrast, the constancy of the ratio for areas of circles is proven as Proposition II of Book XII of Euclid's *Elements*. This despite the fact that measuring, approximately, circumference is simple while it is remarkably hard to measure area.

In terms of rigorous foundations it would be more reasonable to define π as the ratio $\frac{A}{r^2}$ which we can prove using basic geometry!

This brings us to a very interesting question: If $\pi \approx 3.14159$ is the constant for the circumference (if we measure via $C = \pi d$, otherwise the constant is 2π since $C = 2\pi r$), why does this constant have anything to do with the area constant?

7. For each of the three shapes you have chosen to investigate, compare their π -like perimeter constant to their π -like area constant. Are any of these constants the same? Do you see any relationships between the two constants?

Returning to the case of circle, the miracle that the constant in question appears to be π in both cases is something that needs to be proven so we can have certainty.



Sectored Circles

Please use the enlarged copies of the sectored circles above on the last page.

8. Cut out the sectored circle on the left. Try to rearrange the pieces in a way that they create a shape that resembles a shape whose area is easy to determine.

9. Repeat Investigation 8 with the sectored circle in the middle. Try to make an arrangement that is similar for each of these circles.

10. Repeat Investigation 8 with the sectored circle on the right. Try to make an arrangement that is similar for each of these circles.

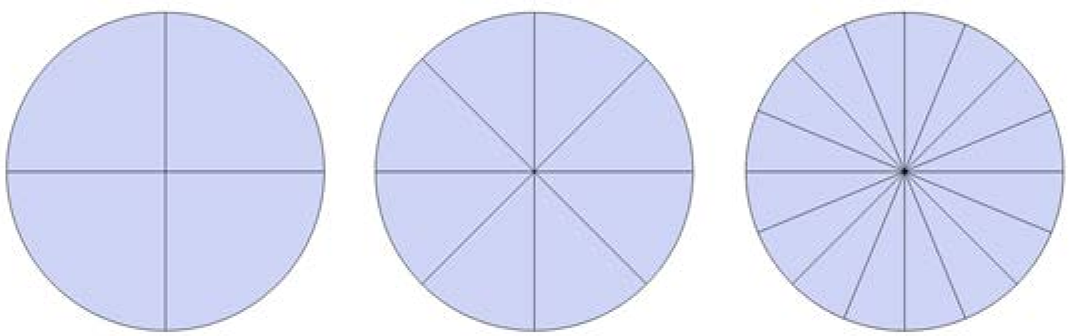
11. Suppose you kept sectoring the circle into more and more congruent sectors. Could you continue to arrange the pieces as you did above?

12. If you continued to do this indefinitely, in the limit what would be the resulting shape?

13. Determine the area of this shape in terms of the original dimensions of the circle.

14. Explain how this establishes a direct link between the perimeter constant and the area constant for a circle.

15. Return your attention to the sectored circles. Can you adapt your strategy to give an alternative explanation why the perimeter and area π-like constants for these shapes are equal?



Sectored circles

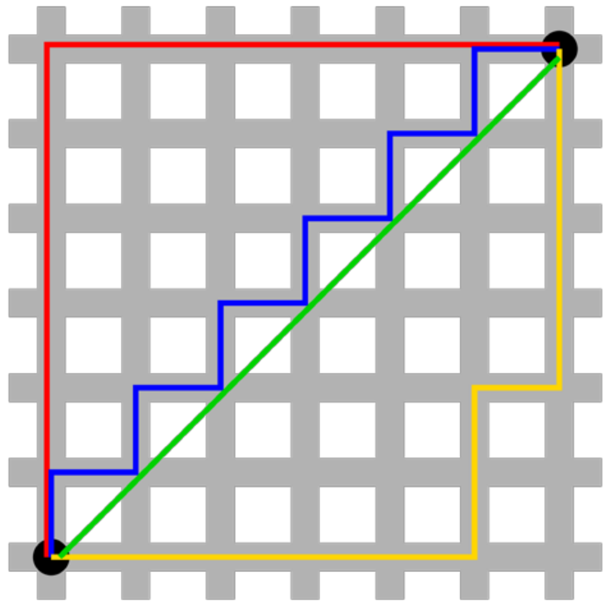
This argument has appeared many times historically, including a seventeenth century Japanese text and the works of Leonardo da Vinci (Italian painter, sculptor, architect, musician, mathematician, engineer, inventor, anatomist, geologist, cartographer, botanist, and writer; 1452 - 1519).[1] It was likely known much earlier than this. It is hard to believe that Archimedes was unaware of it.

We now have a real link which unites the circumference constant for circles back to the area constant. This approach is not typical of the type of geometry practiced by Euclid. Rather, it is typical of that practiced by Archimedes, in ways that foreshadowed the development of calculus almost two millennia later. We now, finally, have proof that the perimeter constant is the same as the area constant - our friend π.

[1] See pp. 17-19 of *A History of Pi* by Petr Beckmann.

Taxicab π

We now consider an alternative geometry. In this new geometry, called Taxicab geometry, distances are measured the way taxicabs drive city whose streets are laid out so they form a regular, square grid. If you wished to travel by cab from one corner to the opposite corner on the same block a cab would have to drive up one block and then over one block. Hence, the distance between these two points is 2. This is a different way to measure distances than in Euclidean geometry. In Euclidean geometry the distance would be, calculated via the Pythagorean theorem, 2. This is the distance the proverbial crow flies. But a taxicab cannot drive diagonally through the center of a block.



16. Give a precise definition of a circle.

17. On square grid graph paper choose and clearly mark an origin. Find and mark a point whose taxicab distance from the origin is 3 blocks.

18. Find and mark another point whose taxicab distance from the origin is 3 blocks.

19. Repeat Investigation 18.

20. Continue to repeat Investigation 18 until you have found all points whose taxicab distance from the origin is 3 blocks. How many such points have you found? In what shape are these points located?

The taxicab geometry we would like to consider is continuous taxicab geometry, where one can follow the streets which make up our grid but can also move along "alleyways" that occur anywhere between the main streets as long as they run parallel to the streets.

21. Utilizing alleyways, find several more points whose taxicab distance from the origin is 3.

22. If you continued using such alleyways, draw the figure which represents all points that are a distance of 3 blocks from the origin.

23. What shape is formed by all of these points?

24. Your shape is formed by all points that are a fixed distance from the origin. Refer to Investigation 16. What do we call such a shape? (Revise your definition, if you wish.)

25. Determine the circumference of your circle of radius 3. Explain carefully how you have determined the circumference.

26. Now draw a circle of radius 2.

27. What is the circumference of this circle?

28. Now draw a circle of radius 1.

29. What is the circumference of this circle?

30. Based on these examples, make a conjecture about the circumference of a circle of radius r in taxicab geometry for positive integers $r > 0$.

31. Prove your conjecture.

32. Having proved the formula for the circumference of circles, determine the value of pi in Taxicab geometry.

Spherical π

You may object that $\pi = 4$ is disingenuous as we are measuring distances so differently in Taxicab geometry. Taxicab geometry is a legitimate geometry. Let us then consider a more natural geometry - the geometry of the surface of the earth. We are, after all, creatures of the earth.

As noted earlier, the relinquishing of Euclid's parallel postulate caused a revolution in geometry in the nineteenth century. Ancient mariners and astronomers did not wait that long to study spherical geometry - they were adept at it close to two millennia earlier.

Find a large spherical object. Lenart spheres are wonderful manipulatives for exploring spherical geometry. If you do not have access, a basketball or other similarly sized spherical object will work fine.

33. On a flat surface, how can you use a piece of string and writing instrument to draw a circle?

34. How do you measure the radius of your circle?

35. You should be able to repeat this same process on your sphere. Draw a circle on the sphere in this way. Does it look like a legitimate circle?

36. Is the string the shortest distance along the surface of the sphere to get from your origin to the circle? If so, does it make sense to call this the radius?

37. Measure your radius and measure the circumference of your circle.

38. Draw another circle on the sphere. Measure its radius and its circumference.

39. Repeat Investigation 38 drawing a circle whose size is much different than those already drawn.

40. Repeat Investigation 38, again trying to draw a circle of a significantly different size.

41. If you let your radius become so large that the circle is in the hemisphere opposite to the "center" where the string is anchored, what happens to the circumference of the circle as the radius gets larger.

42. For each of your circles, compute the ratio of the circumference to twice the radius. This should be our spherical π. Is it? Explain.

But Why 3.14159...?

So in some geometries, shapes have their own πs, often a different π for perimeter than for area. In some they don't.

So the Euclidean circles we are used to have their own π - one you have discovered is the circumference and area constant. But why is this π such a mysterious, complex number? After all, circles are the most symmetric, most perfect of shapes. Why is Euclidean circles' π so esoteric?
The reason is because our way of measuring length is not compatible with the nature of circles. We measure length with straight rulers and square units of area. This is fine for squares and triangles. But our entire measurement apparatus is contrary to the nature of circles. We have a

different paradigm, a different perspective than the circle. The operation of the circle must be translated back into our language of lengths and areas. The translation is complex. It is π.
It should not be any other way, should it? π is the crowning jewel of the beauty of the circle perfectly appropriate.

Our race's efforts to understand π is one of the greatest stories of exploration our history holds. Each culture in each age has worked toward understanding its secrets. And their efforts illustrate the evolution of our ways of knowing. What we have found in each case is that π transcends any finite method of measurement.

Each of the formulas below is a formula for π. Each is listed under the name(s) of the mathematician who we believe is the first to discover it.

Nilakantha Somayaji (Indian mathematician and astronomer; 1444 - 1544)

$$\frac{\pi}{8} = \frac{1}{2} - \frac{1}{4^2-1} - \frac{1}{8^2-1} - \frac{1}{12^2-1} - \frac{1}{16^2-1} - \dots$$

Francois Viete (French mathematician; 1540 - 1603)

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \times \dots$$

John Wallis (English mathematician; 1616 - 1703)

$$\frac{\pi}{2} = \frac{2 \times 2}{1 \times 3} \times \frac{4 \times 4}{3 \times 5} \times \frac{6 \times 6}{5 \times 7} \times \cdots$$

Isaac Newton (English mathematician and physicist; 1642 - 1727)

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{3 \times 2^3}\right) + \frac{1 \times 3}{2 \times 4}\left(\frac{1}{5 \times 2^5}\right) + \frac{1 \times 3 \times 5}{2 \times 4 \times 6}\left(\frac{1}{7 \times 2^7}\right) + \dots$$

Madhava (Indian mathematician; circa 1380 - circa 1420), **James Gregory** (Scottish mathematician; 1638 - 1675) and **G.W. Leibniz** (German mathematician and philosopher; 1646 1716)

$$frac\pi 4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

William Brouncker (English mathematician; 1620 - 1684)

$$\frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \dots}}}}$$

Abraham Sharp (English mathematician; 1653 - 1742)

$$\frac{\pi}{6} = \sqrt{\frac{1}{3}} \times \left[1 - \left(\frac{1}{3 \times 3}\right) + \left(\frac{1}{3^2 \times 5}\right) - \left(\frac{1}{3^3 \times 7}\right) + \dots\right]$$

Leonhard Euler (Swiss mathematician; 1707 - 1783)

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Srinivas Ramanujan (Indian mathematician; 1887 - 1920)

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \left[\frac{(4n)!}{(n!)} \times \frac{[1103 + 26390n]}{(4 \times 99)^{4n}} \right]$$

David Chudnovsky (American mathematician; 1947 -) and **Gregory Chudnovsky** (American mathematician; 1952 -)

$$\frac{1}{\pi} = 12 \times \sum_{n=0}^{\infty} \left[(-1)^n \times \frac{(6n)!}{(n!)^3 (3n)!} \times \frac{13591409 + 545140134n}{640320^{3n+\frac{3}{2}}} \right]$$

David H. Bailey (American mathematician and computer scientist; 1948 -), **Peter Borwein** (Canadian mathematician; 1953 -) and **Simon Plouffe** (Canadian mathematician; 1956 -)
Continued Fraction whose coefficients follow no pattern, like the digits of π

$$\pi = \sum_{n=0}^{\infty} \left[\frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \right]$$

Continued Fraction whose coefficients follow no pattern, like the digits of π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \dots}}}}}}}}$$

Each of these methods involves the infinite.

Current Status of π

Using continued fractions, in 1761 Johann Heinrich Lambert (Swiss mathematician and physicist; 1728 - 1777) was able to prove that π is irrational - it cannot be written as fraction using whole numbers. It is not hard to show that the decimal expansion of any irrational number is nonrepeating. Hence, the decimal expansion of π does not repeat.

Euler and others had long expected that π was even more esoteric, that it was transcendental - not the solution to any polynomial equation with rational coefficients. π resisted for quite some time until Carl Louis Ferdinand von Lindemann (German mathematician; 1852 - 1939) proved that π was transcendental in 1882. This was seen as a remarkable achievement.

In 1995 two remarkable results were obtained, both spigot algorithms for computing the digits of π. Up to that point all efforts to compute digits of π relied on infinite expressions and sophisticated floating point computer arithmetic. The first, by Stanley Rabinowitz (; -) and Stan Wagon (; -), was a method for computing the digits of π one at a time without any reference to previous digits or need for high precision, floating point arithmetic. All that need be specified in advance was how many digits were desired. The second, by David H. Bailey (American mathematician and computer scientist; 1948 -), Peter Borwein (Canadian mathematician; 1953 -) and Simon Plouffe (Canadian mathematician; 1956 -) allowed any single digit of π to be determined directly without any knowledge of previous digits or much significant calculation. The drawback? The digits are computed not as base ten digits but as base sixteen digits. Why does π submit more readily to base sixteen computations? We do not know.
And what about the claim in Takei's meme that the digits of π contain every possible finite string of digits with the expected frequency? Such numbers are called normal. It is unknown whether π is normal. This remains a great mystery. It is a mystery we may never know the answer to.

While the open questions in mathematics far outnumber the questions that have been answered, this is another situation where there is a remarkable silver lining. Mathematicians have proven that the numbers that are normal outnumber those that are not by any measure. Picked randomly, a number is overwhelmingly likely to be normal - to have the remarkable properties claimed in the meme.

3.8.9 You and π

At the outset of this section you were asked some questions about your relationship with π. So how has your relationship with π changed through these investigations?

43. Write a brief essay of one- to two-pages which describes how your relationship with π has changed through the course of these investigations. Some possible topics to include in your essay are:

- What did you learn about π through these investigations?
- What was most surprising to you?
- Were some of the questions you had at the outset answered by your investigations?
- Are you more or less curious about π having completed these investigations?
- Does π make more or less sense to you having completed these investigations? Is this good or bad?