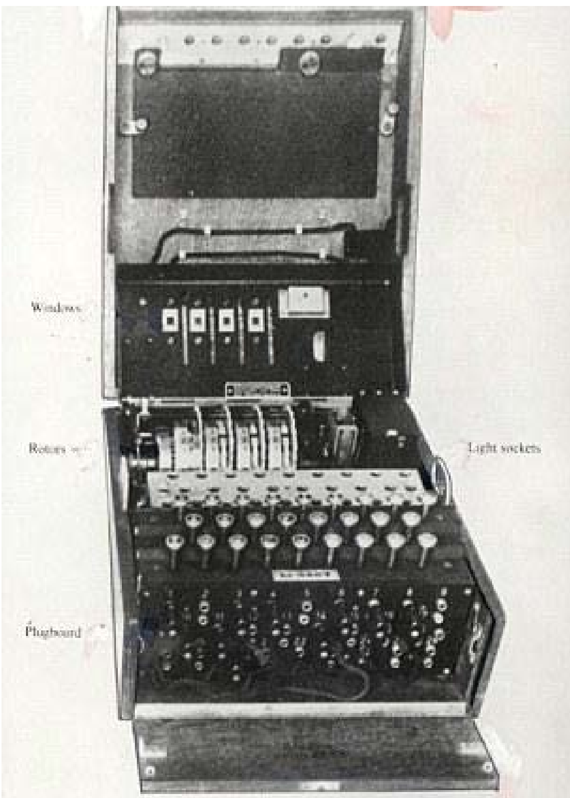
The Language of Logic

Logic is an instrument for analysis of reasoning, whose object is to distinguish good, or valid, reasoning from bad, or invalid, reasoning . It is not a mechanism for determining if particular statements are true or alse; that is a matter for experimentation, consideration; or research. With logic, we consider a collection of supportive statements and ponder the following question: if it is assumed that each of the: collection of statements is true, then does a purported consequent statement have to be true?



You might be wondering what the study of logic would have to do with information technology, but be assured the answer is that the connection between the two is both intimate and essential.

Early in the twentieth century, the work of **Alan Turing** (British computer scientist, mathematician, logician, cryptanalyst and theoretical biologist: 1912 -1954 and **Alonzo Church** (American mathematician and logician ; 1903- 1995 spurred the development of a new field of inquiry re-ferred to as "computability." Turing's inspiration was the desire to solve the decision problem posed by Hilbert, asking whether a standard procedure could be de-veloped to determine whether a particular statement could be proven. This is, as you can appreciate, one and the same as the challenge of whether a particu-lar computer algorithm can be devised to perform a particular task.

Turing's study led to his development of theoreti-cal computation devices now known collectively as Turing machines, which have evolved into what is re-ferred to as a compiler, an essential component for the processing of any programming language. The behavior of a Turing machine is, by construction, based entirely on the applica-tion of logic, and as such devices are the progenitors of the modern computer, it goes without saying that a solid foundation in logic will prove invaluable in your studies.

Mathematical Statements

Three of the most important kinds of sentences in mathematics are universal statements, conditional statements, and existential statements:

A **universal statement** says that a certain property is true for all elements in a set. (For example: *All positive numbers are greater than zero*.)

A **conditional statement** says that if one thing is true then some other thing also has to be true. (For example: *If 378 is divisible by 18, then 378 is divisible by 6*.)

Given a property that may or may not be true, an **existential statement** says that there is at least one thing for which the property is true. (For example: *There is a prime number that is even*.)

Universal Conditional Statements

Universal statements contain some variation of the words "for all" and conditional statements contain versions of the words "if-then." A ***universal conditional statement*** is a statement that is both universal and conditional. Here is an example:

For all animals *a*, if *a* is a dog, then *a* is a mammal.

One of the most important facts about universal conditional statements is that they can be rewritten in ways that make them appear to be purely universal or purely conditional. For example, the previous statement can be rewritten in a way that makes its conditional nature explicit but its universal nature implicit:

If *a* is a dog, then *a* is a mammal.
Or : If an animal is a dog, then the animal is a mammal.

The statement can also be expressed so as to make its universal nature explicit and its conditional nature implicit:

For all dogs *a,* *a* is a mammal.
Or : All dogs are mammals.

The crucial point is that the ability to translate among various ways of expressing universal conditional statements is enormously useful for doing mathematics and many parts of computer science.

Rewriting a Universal Conditional Statement

1. Fill in the blanks to rewrite the following statement:

For all real numbers x , if x is nonzero then x^2 is positive.

a. If a real number is nonzero, then its square _______________ .

b. For all nonzero real numbers x ,______________ .
c. If x ______________ , then ______________ .
d. The square of any nonzero real number is ______________ .
e. All nonzero real numbers have .

Universal Existential Statements

A **universal existential statement** is a statement that is universal because its first part says that a certain property is true for all objects of a given type, and it is existential because its second part asserts the existence of something. For example:

Every real number has an additive inverse

In this statement the property "has an additive inverse" applies universally to all real numbers. "Has an additive inverse" asserts the existence of something - an additive inverse - for each real number. However, the nature of the additive inverse depends on the real number; different real numbers have different additive inverses. Knowing that an additive inverse is a real number, you can rewrite this statement in several ways, some less formal and some more formal:

All real numbers have additive inverses.
Or : For all real numbers r , there is an additive inverse for r .
Or : For all real numbers r, there is a real number s such that s is an additive inverse for r.

Introducing names for the variables simplifies references in further discussion. For instance, after the third version of the statement you might go on to write: When *r* is positive, *s* is negative, when *r* is negative, *s* is positive, and when *r* is zero, *s* is also zero. One of the most important reasons for using variables in mathematics is to refer to quantities unambiguously throughout a lengthy mathematical argument.

2. Fill in the blanks to rewrite the following statement: Every pot has a lid.
   a. All pots ______________ .
   b. For all pots *P*, there is ______________ .
   c. For all pots *P*, there is a lid *L* such that ______________ .

Existential Universal Statements

An **existential universal statement** is a statement that is existential because its first part asserts that a certain object exists and is universal because its second part says that the object satisfies a certain property for all things of a certain kind. For example:

There is a positive integer that is less than or equal to every positive integer.

This statement is true because the number one is a positive integer, and it satisfies the property of being less than or equal to every positive integer. We can rewrite the statement in several ways, some less formal and some more formal:

> Some positive integer is less than or equal to every positive integer.
> *Or* : There is a positive integer *m* that is less than or equal to every positive integer.
> *Or* : There is a positive integer *m* such that every positive integer is greater than or equal to *m*.
> *Or* : There is a positive integer *m* with the property that for all positive integers n, $m \le n$.

3. Fill in the blanks to rewrite the following statement in three different ways:

   There is a person in my class who is at least as old as every person in my class.

   a. Some ______________ is at least as old as ______________ .
   b. There is a person *p* in my class such that *p* is ______________ .
   c. There is a person *p* in my class with the property that for every person *q* in my class, *p* is ______________ .

Some of the most important mathematical concepts, such as the definition of limit of a sequence, can only be defined using phrases that are universal, existential, and conditional, and they require the use of all three phrases “for all,” “there is,” and “if-then.” For example,

> if $a_1, a_2, a_3, \ldots$ is a sequence of real numbers, saying that the limit of an as n approaches infinity is L means that
>
> **for all** positive real numbers ε, **there is** an integer N such that
> **for all** integers n, **if** $n > N$ **then** $-\varepsilon < a_n - L < \varepsilon$.

Logic

1. A universal statement asserts that a certain property is __________ for ______________ .

2. A conditional statement asserts that if one thing _______________ then some other thing _______________ .

3. Given a property that may or may not be true, an existential statement asserts that _______________ . for which the property is true.

In 4 - 9 , fill in the blanks using a variable or variables to rewrite the given statement.

4. Is there a real number whose square is −1?

a. Is there a real number x such that _______________ ?

b. Does there exist _______________ such that $x^2 = -1$?

5. Is there an integer that has a remainder of 2 when it is divided by 5 and a remainder of 3 when it is divided by 6?

a. Is there an integer *n* such that *n* has _______________ ?

b. Does there exist _______________ such that if *n* is divided by 5 the remainder is 2 and if _______________ ?

Note: There are integers with this property. Can you think of one?

6. Given any two real numbers, there is a real number in between.

a. Given any two real numbers *a* and *b*, there is a real number *c* such that c is _______________ .

b. For any two _______________ , _______________ such that $a < c < b$.

7. Given any real number, there is a real number that is greater.

a. Given any real number *r* , there is _____________ *s* ,such that *s* is .

b. For any _______________ , _______________ .such that $s > r$.

8. The reciprocal of any positive real number is positive.

a. Given any positive real number *r*, the reciprocal of _______________ .

b. For any real number *r* , if *r* is , then _______________ .

c. If a real number *r* _______________ , then _______________ .

9. The cube root of any negative real number is negative.

a. Given any negative real number *s*, the cube root of _______________ .

b. For any real number *s*, if *s* is _______________ , then _______________ .

c. If a real number *s* _______________ , then _______________ .

10. Rewrite the following statements less formally, without using variables. Determine, as best as you can, whether the statements are true or false.

a. There are real numbers *u* and *v* with the property that $u + v < u - v$.

b. There is a real number x such that $x^2 < x$.

c. For all positive integers *n*, $n^2 \geq n$.

d. For all real numbers *a* and *b,* $|a + b| \leq |a| + |b|$.

In each of 11 - 16, fill in the blanks to rewrite the given statement.

11. For all objects *J* , if *J* is a square then *J* has four sides.

a. All squares _______________ .

b. Every square _______________ .

c. If an object is a square, then it _______________ .

d. If *J* _______________ , then *J* _______________ .

e. For all squares *J* , _______________ .

12. For all equations *E* , if *E* is quadratic then *E* has at most two real solutions.

a. All quadratic equations _______________ .

b. Every quadratic equation _______________ .

c. If an equation is quadratic, then it _______________ .

d. If *E* _______________ ., then *E* _______________ .

e. For all quadratic equations *E* , _______________ .

13. Every nonzero real number has a reciprocal.

a. All nonzero real numbers _______________ .

b. For all nonzero real numbers *r* , there is _______________ for *r* .

c. For all nonzero real numbers *r* , there is a real number s such that _______________ .

14. Every positive number has a positive square root.

a. All positive numbers _______________ .

b. For any positive number *e*, there is _______________ for *e*.

c. For all positive numbers *e*, there is a positive number *r* such that _______________ .

15. There is a real number whose product with every number leaves the number unchanged.

a. Some _______________ has the property that its _______________ .

b. There is a real number *r* such that the product of *r* _______________ .

c. There is a real number *r* with the property that for every real number s, _______________ .

16. There is a real number whose product with every real number equals zero.

a. Some _______________ has the property that its _______________ .

b. There is a real number a such that the product of *a* _______________ .

c. There is a real number a with the property that for every real number *b*, _______________ .