Applications of Sets

Now that you have developed some of the terminology and concepts of sets, let’s turn our attention to some of the applications in which sets might prove useful.

Applications of Subsets

There are many interesting applications of subsets, some of which are presented in the exercises. One of those, which we will discuss as a means of closing this section, is the voting coalition problem. The premise is that a particular com-mittee or electorate is charged with a task, and the action of the committee is determined by the outcome of a vote in which each member will either have a vote equal in strength to all other voting members or in which the votes are "weighted" in such a manner that some voters have more voting strength than others. Some predetermined quantity of votes, perhaps a simple majority, is required for a decision to be made, and the question might be how many "winning" coalitions exist within the voting structure. Such a question leads one naturally to the subject of probability, which we will investigate later in this text.

For now, let's suppose that four students form the Student Government Association and that one, the president, has two votes, while the other mem-bers have one vote each. If a proposal comes before the association, it is put to a vote by the four students, and a total of three votes (simple majority) is needed for approval of the proposal. How many coalitions of votes will produce approval?

Identify the four students as P (the president), A, B, and C.

16. Con-struct and analyze the set of all possible voting outcomes where a particular individual has voted to approve the proposal. This set would consist of elements such as PAC, which is intended to indicate that the president and students A and C voted in favor.

a) Construct the set of all of the possible voting outcomes:

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b) Construct the subset of the eight winning coalitions of votes that would allow the proposal to be passed by the Student Government Association.

17. For the following problems, determine if the statements are true or false. Explain your answer or give an example to illustrate.

a) There exists a set A such that $A \subset A$.

b) There exists a set A having exactly 20 subsets.

c) If a set A has 40 members and you can list one subset per second and work continuously for seven days, it is possible to list all the subsets of A in that time.

d) If A and B have the same cardinality, then so do 2^A and 2^B.

e) $\varnothing \subset \varnothing$

f) For sets A, B, and C, if $A \subset B$ and $B \subset C$, then it must be the case that $A \subset C$.

18. If a sandwich shop allows you to pick any combination of ingredients from its menu to build a Panini sandwich and the menu shows 18 ingre-dients, how many different sandwiches are possible?

19. Four voters are going to vote yes or no (Y or N) on an upcoming town council issue. The measure will pass if three or more of the voters vote yes. How many different voting combinations exist, and how many of these will be results that allow the measure to pass?

20. A tyrannical historical commission has the power of decision over per- mission for home-owners in a neighborhood to repaint their homes a proposed color. The commission has four members, each having a par- ticular number of votes: president (four votes), vice president (three votes), architectural committee leader (two votes), and fence height overseer (two votes). If a simple majority of votes is needed to approve a resident's application, how many winning "coalitions" of votes exist?

The Survey Problem

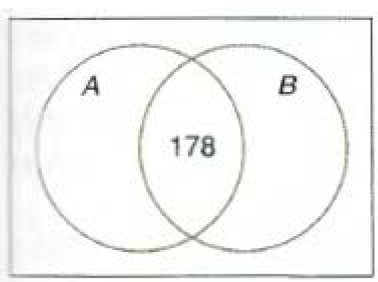
In a survey problem, we have information about the cardinality of various sets and their intersections or unions with other sets. The problem is, typically, to determine some fact about the situation, such as the total number of individu-als involved or the number of individuals possessing some particular trait. Such problems are often encountered in the leisure press in the form of "logic puzzles”

Consider the following situation. A group of 500 students are asked if they are taking a math course, a philosophy course, both, or neither. The poll finds that 178 students are taking both types of courses, 88 students are taking neither type of course, and 308 are taking a philosophy course. With that information in hand, we are now asked the following questions: How many students are taking only a philos-ophy course but not a math course? How many students are taking a math course? How many students are taking *only* a math course but not a philosophy course?

You have probably seen such problems in the past and recognize that it boils down to assembling the data in an organized fashion and using the known infor-mation to piece together the facts you require in order to answer the questions.

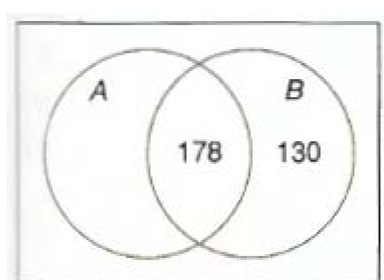
The process can be systematized by construction of a Venn diagram, as we will see, and this is particularly useful when the problems become more complex, as will be the case in the next example.

There are two sets involved here: A = {students taking a math class}, and B = {students taking a philosophy class}.The universal set is U = {500 students being surveyed}. We will construct a two-set Venn diagram and use the given information to fill in the various components of the diagram with the cardinality of the set depicted by each region. It is typically best, if possible, to start in the middle of the diagram and work your way outward, but it is not necessary, and we may not have enough information to do so.

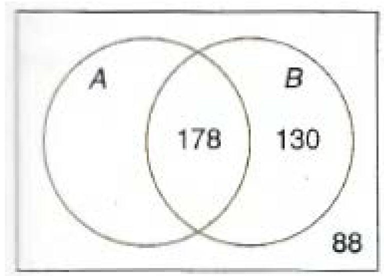


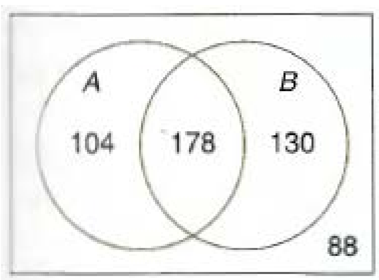
We'll start with the general diagram (see figure on the left) and observe that we are told that 178 students are taking both course types. The diagram will be gradually "filled in" with the information as we consider it.
There is no specific series of steps we must follow inorder to reach our conclu-sion, since every problem will provide us with different combinations of informa-tion. For this reason, filling in the rest of the diagram must be performed in an ad hoc manner where we attempt to deduce the cardinality of each region of our graph.

We note that the problem told us 308 students were taking a philosophy course. Since 178 students are already accounted for in the overlap of the math and philosophy sets, this implies that 130 students must be within the philoso-phy class circle yet outside the overlap with the math class circle. Thus, we can fill in a second part of the diagram, as shown on the right.



Finally, we are told 88 students took neither course and thus lie within the universal set but outside of both of the individual sets, as seen below:





There is just one region of the Venn diagram whose cardinality is yet to be determined. This is the group of students taking a philosophy course. The problem gives no direct data about that collection of students, but we can make an inference based on all available data. The universal set was known to have 500 members, and 396 are now accounted for in the already-known areas. This means 104 students are unaccounted for and therefore must lie in the remaining region.

The situation is slightly more complicated when three sets are involved but not oppressively so. We will consider an example of such a problem before look-ing at a different sort of application.

1. Suppose 200 households are surveyed to determine how many cats, dogs, and ferrets are kept as pets. It is found that three households have all three types of pet, five have a cat and a ferret, 11 have a dog and a ferret, 20 have a cat and a dog, 66 have only a cat, 46 have only a dog, and 46 have no pets at all. Construct a Venn diagram to describe this situation.

Establishment of Correct Reasoning

Venn diagrams can also be used to assess reasoning. The term "reasoning" here refers to a determination if a series of evidential facts necessarily leads to a par-ticular conclusion. When used in this manner, the Venn diagram is referred to as an Euler diagram, named after the famous Swiss mathematician Leonhard Euler (pronounced "oiler").

As an illustration, consider the following series of statements:

- All bachelors are reclusive.
- Some reclusive people are strange.
- Therefore, some bachelors are strange.

Does this sound convincing to you? The statements are rather bold ones, making what we call "universal" and "existential" claims about persons and drawing a con-clusion based on them. The term "universal" refers to the statement employing the word "all" and indicates that the assertion holds for every single bachelor. The term "existential" refers to the statements having the word "some" and purports that there are at least a few people who are strange and a few bachelors who are as well.

The reasoning is what we may describe as "sound" (or "valid" if we can establish that); assuming that we accept the first two statements (which we will come to know as **premises**) as being true, then the third statement, following the word "therefore" must be true as well. That final statement we will refer to as the **conclusion** of the argument.

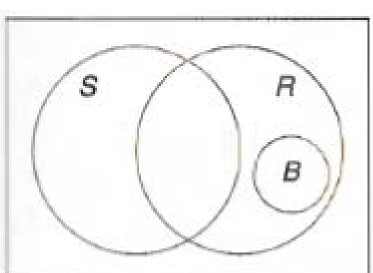
We draw an Euler diagram that captures the precise meaning of the premises and then assess whether the resulting diagram sup-ports the conclusion. Recall that the premises were as follows:

- All bachelors are reclusive.
- Some reclusive people are strange.

We have three types of people being described: bachelors, reclusive people, and strange people. We will name those sets as B, R, and S so that the names of the sets reflect their membership. This is a common practice, to choose set names in such a way that the name of the set relates directly to the membership. It is not necessary, however, and you could always use the names A, B, and C for your sets if you wish. The universal set is not specified, but we can assume it to be any convenient superset of all three sets described in the problem. We will take the superset to be the set of all male humans and designate that as U.

Consider the first statement. "All bachelors are reclusive" tells us that the set of bachelors is entirely contained within the set of reclusive people. ie. a subset. Our experience tells us that some recluses are married, and therefore the set of bachelors must, in fact, be a proper subset of the set of reclusive people.
The second premise, "Some reclusive people are strange" indicates that there exists an overlap between the set of reclusive people and the set of strange people but that not all reclusive people are within the set of strange people. Remember, the premise merely states that *some* reclusive people are strange.

The logical argument is valid if there is no possible way in which we could pro-duce an Euler diagram in contradiction with the content of the conclusion, "Some bachelors are strange." The conclusion indicates that there is definitely an overlap between the set of bachelors with the set of strange people. Must this always be the case?



We can envision an Euler diagram construction where the premises would be satisfied, but the conclusion is not required to be true. Consider, for instance, the diagram at the left. Here, the set of bachelors is clearly a subset of the set of reclusive people, and there are some reclusive people who are within the set of strange people. Notice, though, that there is, in this orientation, no overlap between the set of bachelors and the set of strange people.

You may object that the orientation we have depicted is not the only possibility and that it is conceivable that the circle depicting set S could overlap with set B. This is precisely the point, however! The point is that it is not necessary that an overlap between sets B and S exists, and therefore we say that the argument lacks validity and is disproven using an Euler diagram analysis.

Let's take a look at another example. ".All children are graceful. Sarah is not graceful. Therefore, Sarah is not a child:' Keep in mind that what we are not do-ing is assessing the truth or falsity of the individual statements. It is not relevant, in this situation, whether the statements are factually true. Our question is this: if we assume that the premises are true, must the conclusion be true as well, or is it conceivable that under the given assumptions the conclusion could be false?

We will assign names to the sets involved: C will be the set of children, G the set of graceful people, and S the singleton set consisting of Sarah. The universal set U shall be the set of all people.

The first premise, ".All children are graceful;' indicates that Cis a proper subset of set G. The second premise, "Sarah is not graceful;' tells us that the singleton set S is not contained in the set G and so is in G'. The Euler diagram for this situation strictly limits our positioning of the set S in such a way that it must lie outside the set G and therefore at a distance from the set C. Consequently, it is impossible that Sarah could be a child, and the argument is valid.

Arguments of the form demonstrated in the previous two examples are re-ferred to as syllogisms, which are sets of premises followed by a conclusion. They are special forms of arguments that contain quantifying words, such as some, all, and none.

2. Determine whether the following syllogism is valid: All ID cards are made of plastic. My credit card is made of plastic. Therefore, my credit card is an ID card.

Problem Solving

3. UConn conducts a survey to determine where students are residing. Of 175 students surveyed, it was found that 79 lived on campus, 93 lived in an apartment, and 44 lived in apart-ments on campus. Of those interviewed, how many lived in apartments off campus? How

many lived on campus but not in an apartment? How many students lived neither on campus nor in an apartment?

4. In a small midwestern town, a survey was conducted of the 125 residents of a particular neighborhood. It was found that 88 of the residents owned a car, 59 owned a truck, and 21 owned no vehicle at all. Of those sur-veyed, how many owned both a car and a truck? How many owned only a truck? How many owned only a car?

5. At a particular cat rescue, a set of 50 cats was examined. It was found that 12 had only black fur, 11 had only orange fur, 9 had only gray fur, 24 had some black fur, 6 had some black and some orange fur but no gray fur, 5 had some black fur and some gray fur but no orange fur, and no cats had both gray and orange fur but no black fur. Find the number of cats who had at least some one of the three colors in their fur, the number that had all three colors in their fur, the number that had none of the colors in their fur, and the number that had exactly two of the colors in their fur.

6. A police officer studied 63 cars passing the local high school. He ob-served that 19 cars were driven by what appeared to be teenagers, 37 were driven by females, 13 were driven by what appeared to be teenage boys, 6 were driven by what appeared to be teenage girls, and 31 were driven by women who were past teenage years. How many drivers appeared to be men who were past teenage years?

7. A State Police officer sampled cars crossing the border from Connecticut into Massachusetts. In his report he indicated that of 95 cars sampled, 45 cars were driven by men, 63 cars were driven by Connecticut residents, 53 cars had three or more passengers, 37 cars were driven by men who were Connecticut residents, 35 cars were driven by men and had three or more passengers, 30 cars were driven by Connecticut residents and had three or more passen-gers, and 25 cars were driven by men who were Connecticut residents and had three or more passengers. His supervisor read the report and deter-mined that the report was in error. Explain how the supervisor knew this.

8. A survey was taken of students at WCSU. If 75 students were sur-veyed and it was found that 25 were majoring in engineering, 26 were majoring in biology, 30 were majoring in chemistry, 8 were double majoring in engineering and chemistry, 11 were double majoring in en-gineering and biology, 7 were double majoring in biology and chemistry, and no one was triple majoring. How many students were majoring in engineering only, biology only, and chemistry only?

9. A survey of 427 farmers showed that 135 grew only beets, 120 grew only radishes, 100 grew only turnips, 210 grew beets, 50 grew beets and rad-ishes, 45 grew beets and turnips, and 37 grew radishes and turnips. Find the number of farmers who grew at least one of the three, grew all three, did not grow any of the three, or grew exactly two of the three.

10. In Danbury, 150 children were surveyed, and it was found that a total of 35 played basketball, 71 played baseball, 30 played soccer, 10 played all three, 3 played only soccer, 17 played only basketball, 6 played only soccer and basketball, 48 played only baseball, and 53 played none of the three sports. Find how many played soccer and baseball only, and how many played basketball and baseball only?