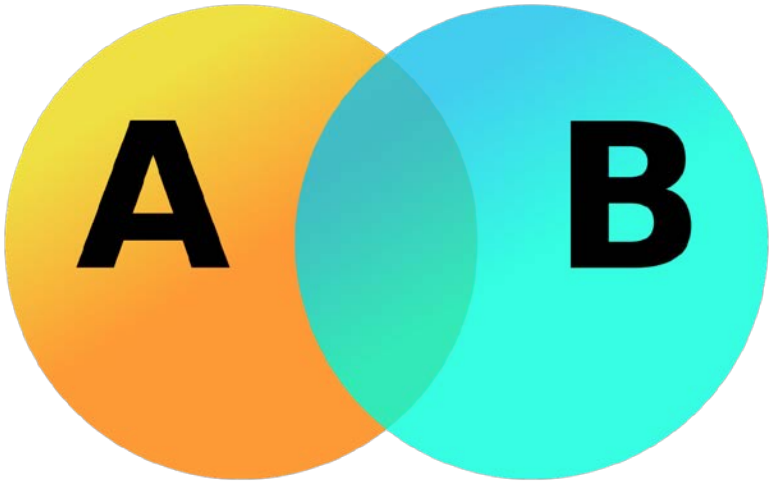
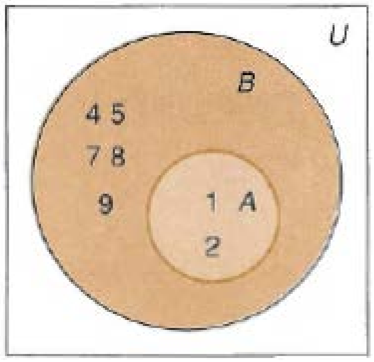


Venn Diagrams

There are times when it proves useful or desirable for us to represent sets and the relationships among them in a visual manner. This can be beneficial for a variety of reasons, among which is the possibility that a pictorial representation may reveal relationships that were unclear through other descriptive styles.

The method we are going to use is due to John Venn and was named in his honor. The process is called a **Venn diagram** construction. A Venn diagram uses a series of closed curves, usually circles or ellipses, to depict sets. The circles or el-lipses are placed within a rectangular box that is intended to depict the universal set, and the relationships (overlapping, nonoverlapping, or inclusion) between the circles/ellipses will indicate corresponding relationships among the sets.



The figure on the right shows a typical Venn diagram representing two sets, *A* and *B*, within an arbitrary universal set, *U*. In this situation, the set A = {1,2}, and the set B = {1,2, 4, 5, 7, 8, 9}.

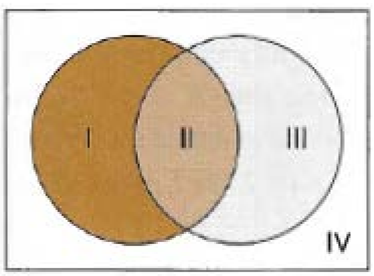
There are two significant features of this particular Venn diagram that we should observe and remark on. The first is that the elements 1 and 2 are possibly not obvious as.members of set B. This is somewhat disguised by the fact that they lie within the circle depicting set A, but since that circle is contained wholly within set B, the elements are also members of *B*. The second observation is that there are no indicated elements outside of set B, and that would suggest that *B* contains all elements of the universal set.

The relationship between the two circles exhibits the same status shared between the sets *A* and *B*. *A* is a proper subset of *B*, since *B* contains all the ele-ments of *A* as well as other elements not in *A*. Note that the circle depicting *A* is contained entirely within the circle depicting B but that there are other ele-ments within the *B* circle and outside the *A* circle.

1. Construct a two-set Venn diagram that depicts the following:
   A = {2, 8, 10}
   B = {2, 4, 6, 8, 10, 12, 14}
   U = {2, 4, 6, 8, 10, 12, 14, 16}

The Intersection of Two Sets

The circles representing two sets within a Venn diagram can appear in a variety of configurations. The circles may be positioned in such a way that one is entirely contained within the other, as you saw in the first example. The two circles may have a partial overlap, or they may not overlap at all.



In the case where the circles had a partial overlapping, the situation would look something like the figure at left. Note that the sets have not been identified; this is only a pictorial convenience to avoid overly cluttering the diagram with additional letters.

The center part of the diagram, labeled with Roman numeral II, represents the members that are common to both sets, and this is an informal definition of **intersection** . Let's formalize this notion, along with the symbol for the intersection relation:

> **Definition**: $A \cup B$, the intersection of sets *A* and *B*, is the set of all elements that are simultaneously elements of set *A* and set *B*.

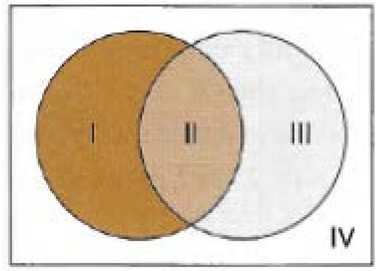
In what situation might sets "overlap" in this manner? Think of any two orga-nizations to which you belong; examples might be your math class and your family. The set *A* might be the set of all students enrolled in your math class and the set B the set of all persons in your family. The sets *A* and *B* have at least one common member: you. Thus, there is an overlap, or intersection, of the two sets that we would describe as the set of all people who are both in your math class and in your family.

2. Construct a two-set Venn diagram that depicts the following:
   A = {my family}
   B = {my math class}

3. Let *A* = the set of all whole numbers less than or equal to 10 and *B* = the set of even integers. Find the intersection of sets *A* and *B*.

4. By using a Venn diagram, produce the prime factorizations of the numbers 42 and 30, and from that Venn diagram determine the greatest common factor of those numbers.

Union of Two Sets



Returning to the matter of Venn diagrams, you can also view the diagram as a way to depict the result of combining the two sets together to form one larger set. When the two sets are combined, we say that we are creating the **union** of the two sets; we obtain the combination of the regions marked I, II, and III on the diagram.

> **Definition**: $A \cap B$, the union of the sets *A* and *B*, is the set of all elements that are elements of set A, set B, or both. Building on our previous example of your class and your family, the union of the two sets would be the set of all the members of your math class, together with the members of your family.

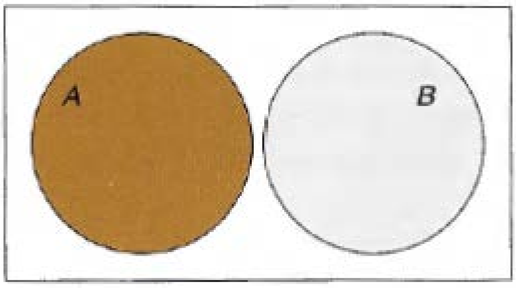
The understanding in mathematics and in logic is that the word "or" is meant to represent the inclusive or, which means "either or both:' So, when we say that x is an element of set *A* or set *B*, then this accepts the possibility that x might be an element of both of the sets as well.

Your Turn:

5. By using a Venn diagram, produce the prime factorizations of the numbers 42 and 30 and use them to determine the least common multiple of 42 and 30.

Disjoint Sets

Another possible orientation of two sets in a Venn diagram would be the case where the two sets were situated in such a way that there was no overlap of their representative circles. In such a case, we say the intersection of the two sets is empty or that the sets share no common members. The two sets are said to be **disjoint**, and the circles depicting them in the Venn diagram would have no overlap, as seen below:



An example of disjoint sets might be the set *A* = all even natural numbers and the set *B* = all odd natural numbers. Since nothing is both an even natural number and an odd natural number, the two sets are disjoint.

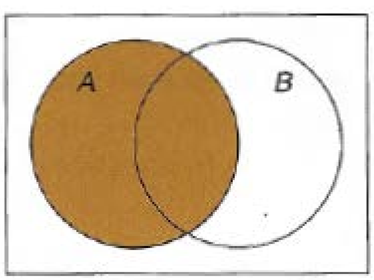
Set-Theoretic Expressions and DeMorgan's Laws

Venn diagrams have another useful function, the confirmation of equality for set-theoretic expressions. When we use the term "set-theoretic expression;' we refer to expressions where sets are associated using the operations of intersec-tion, union, or complementation. An elementary expression with which we are already familiar would be $A \cap B$.

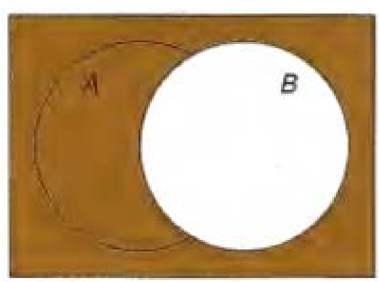
If you were given two set theoretic expressions, such as $A \cap B^C$ and $A \cup (B \cap A)^C$, you might wonder whether they are they equal to one another. There are various methods we could employ to assess the situation, but one useful and timely method employs Venn diagrams. The statements are equal if their Venn diagram representations are identical.

> **Definition**: Two set-theoretic expressions are considered equal if their Venn diagram representations are identical.

Let's consider the first statement, $A \cap B^C$. We want to take the intersection of A with the complement of B, so we consider the two independently.

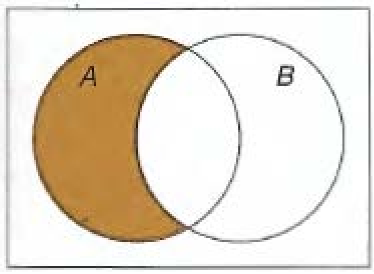


A

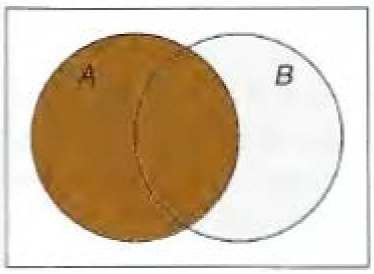


B ^{C}

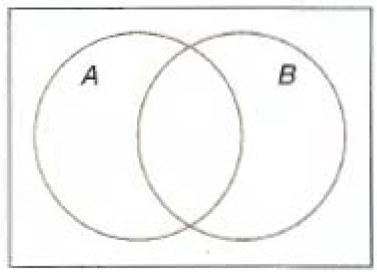
The overlap of the two regions is the part of the Venn diagram shaded simultaneously in both graphs, as shown below:



Now consider the other expression, $A\cup(B\cap A)^C$ and repeat the process. Recalling that $B\cap A$ is the "football-shaped area" where the circles overlap, you can see that its complement is everything outside that part of the diagram.

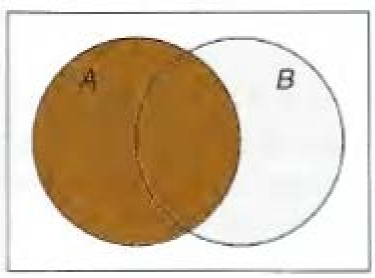


A



$(B\cap A)^C$

When combined, the following results:



Therefore, the two expressions do not yield the same Venn diagram, and hence the expressions are not equal to one another.

There are two famous set-theoretic results we shall ask you to confirm in the exercises named in honor of the logician who first defined them. **DeMorgan's laws**, first formalized by Augustus DeMorgan in the nineteenth century, state that the operations of union and intersection interchange under the operation of complementation . Symbolically,

$(A\cup B)^C = A^C\cap B^C$, and $(A\cap B)^C = A^C\cup B^C$

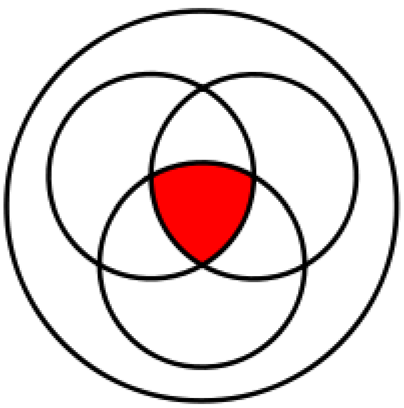
The result was actually known in antiquity (similar results were employed, for example, by Aristotle), but through what is known as the development of algebraic logic, performed by the logician George Boole in the nineteenth century, it became and is still credited to DeMorgan .

Your Turn:

6. Use Venn diagrams to determine if the following are true or false.

   a) $A^C \cap B = B$

   b) $(A \cup B) \cap A = A$

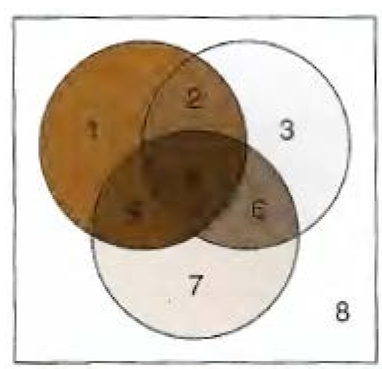


More Than Two Sets

In the examples we have seen so far, we have limited our Venn diagrams to have two sets depicted within the universal set. This is not necessary, however, and there could be as many sets as desired. For practical reasons, we will limit our examples to those having three sets within them. Such a Venn diagram appears on the left.

The situation becomes slightly more complicated with three sets in the Venn diagram, but there are no real additional concepts involved.

Your Turn:



Let the universal set be the set of students at a local high school.

Set A repre-sents the students enrolled in an astronomy class
Set B represents the stu-dents enrolled in an biology class
Set C represents the students enrolled in a chemistry class.

7. Using the labels we had placed on the Venn diagram at the right, interpret each the regions labeled on the diagram. (One example is provided below.)

   a) Region 1:

b) Region 2: *The students in this region are taking both as-tronomy and biology, but are not taking a chemistry class.*

c) Region 3:
d) Region 4:
e) Region 5:
f) Region 6:
g) Region 7:
h) Region 8:

Intersections, Unions, and Cardinality

Whenever we contemplate sets or parts of sets, it's natural to wonder about their cardinality.What about the cardinality of one of the combinations of sets, such as the intersection of two sets, or the union of two sets?

In order for us to understand this, let's consider cases. If the two sets are disjoint, then clearly the cardinality of their union must be the sum of the individual cardinalities as a consequence to the definition of disjoint sets. If they are not disjoint, then we can reason as follows. Suppose the cardinality of set *A* was added to the cardinality of set *B*. We would obtain a total that was in excess of the cardinality of the union because those elements that were in both sets would have been counted twice. In order to compensate for this, we can subtract the cardinality of the intersection, which will remove the double counting. That is,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

We could reason in a similar manner to obtain a formula for the cardinality of the intersection, but there is no need. The formula given above, being alge-braic and involving the cardinality of the intersection as it does, can be solved for that cardinality, and our second formula will be obtained. That is, the for-mula for the cardinality of the intersection of sets A and B is given by

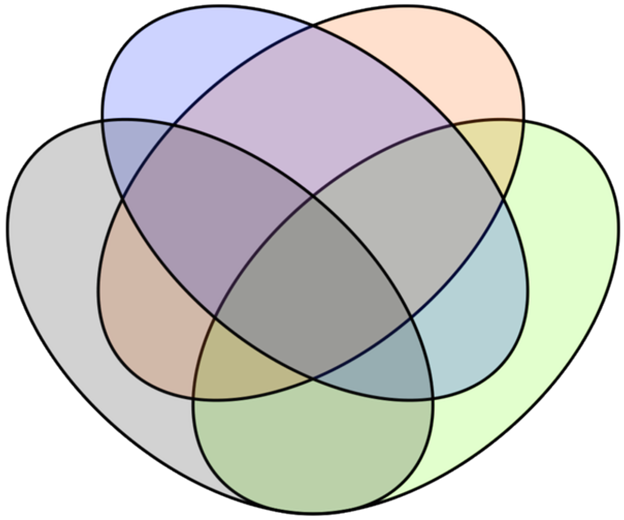
$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

Your Turn:

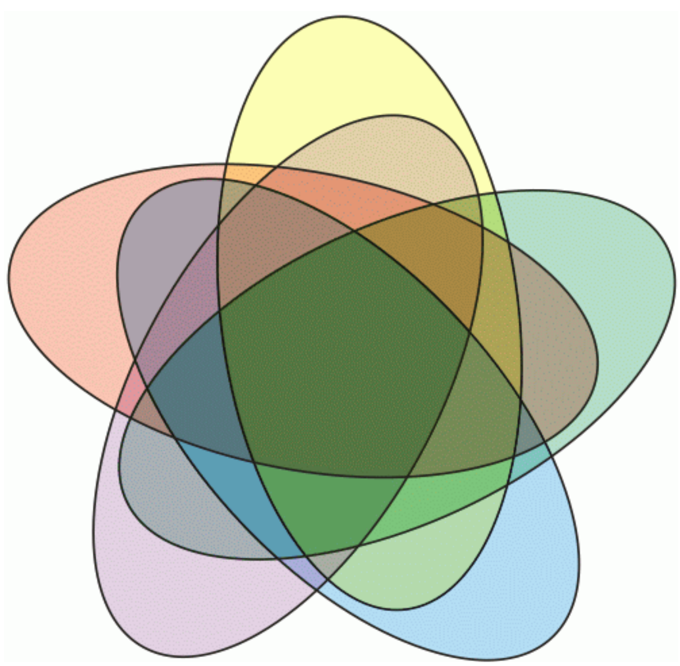
8. Given A = {1,3, 5, 7, 9}, B = {2, 3, 4, 5}. Verify the formulas given for the cardinality of the intersection and of the union by producing the relevant sets and determining their cardinality directly.

Even More Sets

You can use any shapes, in principle, to depict the sets, but the common practice is to use circles or ellipses. When there are more than three sets, the circular approach is generally abandoned, and we appeal to ellipses, which al-low us to fit more sets into our diagram in an aesthetically pleasing manner.



The images generated by Venn diagrams with more than three sets dis-play a feature that is, perhaps, not so evident with the diagrams having only two or three sets. By constructing the diagram using ovals of similar size and rotation determined by dividing 360° by the number of sets, we generate beau-tiful images possessing rotational symmetry.



9. In each of the following, construct a two-set Venn diagram that depicts the described region.
$A \cap B^C$
$A^C \cup B$
$(A \cap B)^C$
$(A \cap B)^C \cup B$
$(A \cup B^C)^C$
$A^C \cap A$
$(A^C \cap A)^C$

10. Use Venn diagrams to determine if the following equalities are true or false.

1. $(A^C \cup U)^C = A$
2. $(A \cap B)^C = A^C \cup B^C$
3. $(A \cup B)^C = A^C \cap B^C$
4. $(A \cup B)^C = A^C \cup B^C$

Problem Solving

For the following sets of Venn diagrams, use set notation to describe the shaded area, using in-tersection, union, or complementation if needed. The answers are not necessarily unique.

11. In the readings of this section, we argued for the formulas for the cardinality of the intersection and for the union of two sets. By similar process of rea-soning, generate the formula for the cardinality of the union of three sets.

12. Following up on Investigation 11, generate the formula for the cardinality of the intersection of three sets.