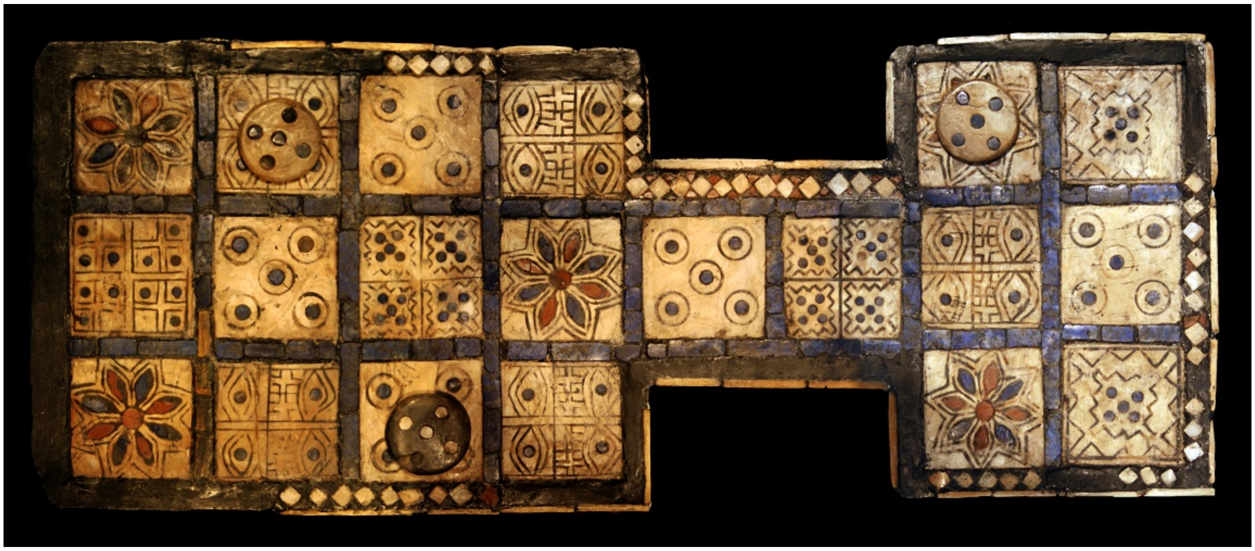


The Hackenbush Game

> I love mathematics not only for its technical applications, but principally because it is beautiful; because man has breathed his spirit of play into it, and because it has given him his greatest game - the encompassing of the infinite.
>
> **Rosza Peter** (Hungarian mathematician;1905 - 1977)

The surreal numbers were invented in the early 1970's by John Horton Conway (English mathematician; 1937 -). Conway is a powerful and prodigious mathematician. One of his areas of specialty is games. Shortly we will see a wonderful connection between the surreal numbers and games.

But first a note about how the world learned about surreal numbers is an order. Conway ate lunch with Donald Knuth (American computer scientist, mathematician; 1938 -) one day shortly after he invented the surreal numbers. Knuth was clearly interested in the topic as less than a year later he decided to explore these numbers in detail. While he did the research needed to understand this number system he began thinking about how one might teach others about these numbers. He decided to write a novelette, entitled Surreal Numbers: How Two Ex-students Turned on to Pure Mathematics and Found Total Happiness, about how people would go about developing such a theory. In the postscript he says:

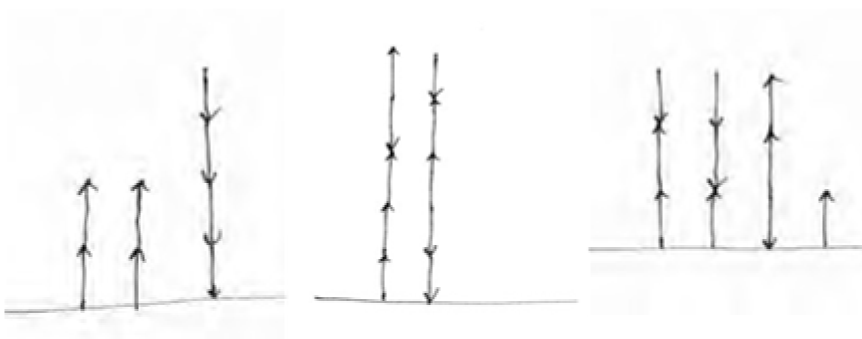
> As the two characters in this book gradually explore and build up Conway's number system, I have recorded their false starts and frustrations as well as their good ideas and triumphs. I wanted to give a reasonably faithful portrayal of the important principles, techniques, joys, passions and philosophy of mathematics, so I wrote the story as I was actually doing the research myself.

Hmmm...Although our book is not a novelette, we hope that some of these things have characterized your exploration through this book; that you have experienced frustrations as well as good ideas and triumphs; felt joy and passion.

This book, this novelette, a piece of literature, was in fact the first way in which the world learned about surreal numbers. Up to the point of its publication, this system had not appeared elsewhere in print!

The Hackenbush Game

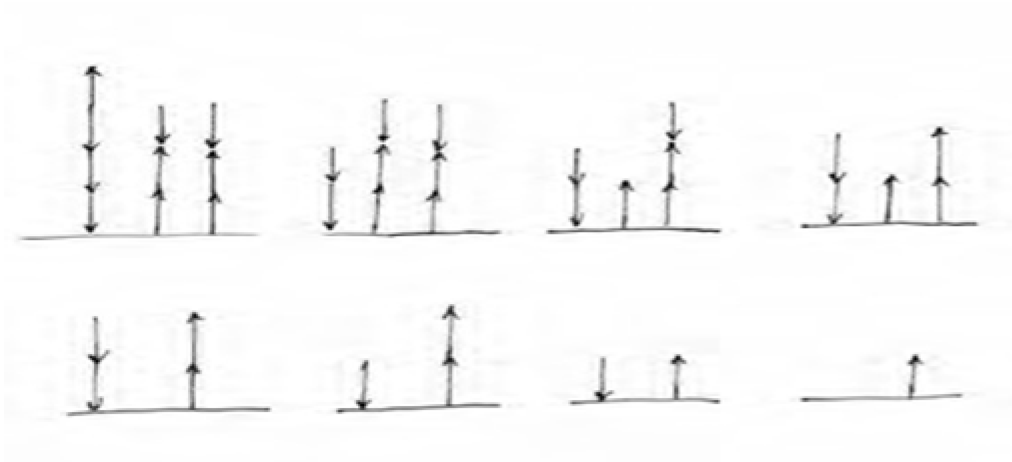
Hackenbush is a game played by two players. Usually the players and Red and Blue, the line segments are colored red and blue, and the line segments can be oriented in any desired way. Here, since we are primarily interested in the connection to surreal numbers, our players will be Up and Down. The playing board for a Hackenbush game, also called **beginning position**, is a collection of line segments made up of ↑'s and ↓'s. The only condition on these segments is that from any point on any segment you can eventually reach the ground by travelling continuously along the arrows. I.e. no arrow can be disconnected from the ground.



Several beginning positions are shown above.

To play the game players take turns removing one of their arrows. They can remove any single arrow that they want. When this arrow is removed, all arrows that are no longer connected to the ground are also removed. Turns alternate. If a player cannot move on their turn (because they have no arrows left), they lose.

A sample game is shown below. Here Down begins and in the last frame Up will remove their ↑. Because there are no ↓'s left, Down loses as they have no move.



A sample Hackenbush game

1. Choose an opponent. Use the second beginning position shown to play a game of Hackenbush. Record your moves and the winner.

2. HackenbushGame1Switch Now switch roles - let the person who played second play first. Now replay the game from Investigation 1. Was the outcome the same or different?

3. Repeat Investigation 1 and 2 for the third beginning position shown.

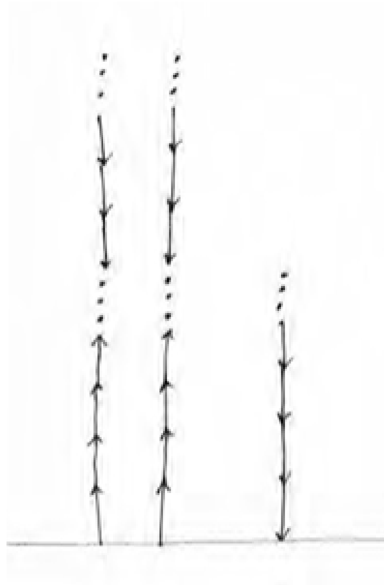
4. Now make up your own beginning position and play another game.

5. Have your opponent make up a beginning position and play another game.

6. Are you starting to get an idea how the strategy of this game works? Explain.

So what is the connection to surreal numbers? In playing the game in Investigation 1 you should have seen that the second player to play was always going to be the winner. When this happens this is called a **null Hackenbush game**. Let us think of what the arrows in the beginning board for this null Hackenbush game represent in surreal terms. The first chain represents the surreal number $\uparrow\uparrow$ which is the integer 2. The same is true for the second chain. The third chain represents the surreal number $\downarrow\downarrow\downarrow\downarrow$ which is the integer -4. The fact that this is a null Hackenbush game means, in surreal terms, $\uparrow\uparrow + \uparrow\uparrow + \downarrow\downarrow\downarrow\downarrow = 0$. In other words this Hackenbush games proves the surreal analogue of the equation $2 + 2 + (-4) = 0$ which we normally read as $2 + 2 = 4$.

This is not very exciting with finite numbers where we are used to the arithmetic already. What's interesting is to use Hackenbush games for surreal numbers that are uniquely surreal.
In each of the examples below we have infinitely many arrows, denoted as usual with an ellipsis. When you play the Hackenbush game and remove an arrow, you must explain precisely what arrow you have removed. E.g. your move can be to remove the 37^{th} arrow.

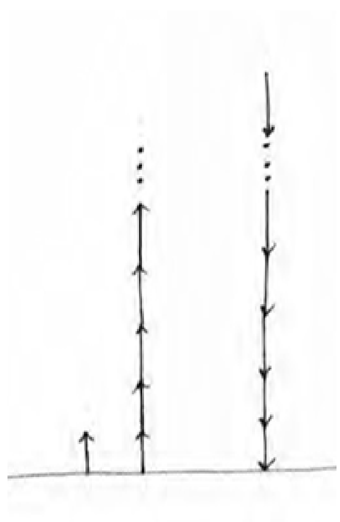


Hackenbush beginning positions with −ω

7. Play the Hackenbush game above several times. Is this a null Hackenbush game?

8. What surreal arithmetical statement does this Hackenbush game correspond to? Explain.

9. From this arithmetical statement, what surreal number do you think ↑↓↑↓↑↓ ... represents? Explain.



10. Now play the Hackenbush game below several times. Is this a null Hackenbush game?

11. What surreal numbers are represented by this Hackenbush game? Explain.
12. What arithmetical statement would this correspond to if this was a null Hackenbush game?

13. What do Investigation 10 through Investigation 12 tell you about addition with surreal numbers?