The Ordered and Surreal Landscapes of Infinity

> The transfinite numbers are in a certain sense themselves new irrationalities and in fact in my opinion the best method of defining the finite irrational numbers is wholly similar to, and I might even say in principle the same as, my method described above of introducing transfinite numbers. One can say unconditionally: the transfinite numbers stand or fall with the finite irrational numbers; they are like each other in their innermost being; for the former like the latter are definite delimited forms or modifications of the actual infinite.
>
> **Georg Cantor** (German mathematician;1845 - 1918)

What's Next?

As we've seen, Cantor's theory of cardinal numbers is based in large part on a playful, almost childlike use of matching to determine "how big?" It is a basic idea which Cantor followed logically to its extreme, into the realm of the transfinites.

In considering the same natural numbers 1,2,3,... the notion of "what's next?" is an equally basic idea. This insight was clear to Cantor who extended this idea to its logical extreme much as he had the notion of "how big" to create the cardinal numbers. The result was the birth of another system of numbers, the ordinal numbers, which we will consider now.

The Ordinal Numbers

The natural numbers 1,2,3,... are endowed with a natural order. Each number is the immediate predecessor of the number which follows it. We have written them in this natural order from left to right. Of course, this is an order that continues indefinitely. At any point in the order the child's injunction "plus one" takes us to the next natural number in the order. From 9 we move to 9 + 1, a.k.a. 10.

But we need not think of 10 only as the number immediately following 9; its immediate successor. We can also think of 10 as the immediate successor of the whole collection {1,2,3,4,5,6,7,8,9} simultaneously. This observation launches us into the transfinite realm once again, for we think beyond the immediate successor of any finite collection of natural numbers to the immediate successor of the entire collection; $N = \{1,2,3,...\}$. Of course this immediate successor, which Cantor denoted by ω, will be infinite because it succeeds every finite number. ω is called the first transfinite ordinal.

But then, having built a number system on the notion of successors, ω would again have an immediate successor itself – a "plus one." Naturally this is $\omega + 1$. And the immediate successor of $\omega + 1$ is, of course, $\omega + 2$. Then $\omega + 3$. And $\omega + 4$. And so on indefinitely. Thus we have the ordinal numbers:

$$1, 2, 3, \ldots, \omega, \omega + 1, \omega + 2, \omega + 3, \ldots$$

What lies beyond the last ellipsis? These transfinite ordinals are followed by many, infinitely many, other transfinite ordinals. We have $\omega+1$, $\omega+2$, $\omega+3$, $\omega+4$,... What could the immediate successor of this collection of ordinals be? Since each of these numbers is simply ω plus a natural number, the immediate successor should be ω plus the immediate successor of all the natural numbers. But this is just ω. Thus, the immediate successor of the ordinals $\omega+1,\omega+2,\omega+3,\ldots$ is the transfinite ordinal $\omega + \omega$. Certainly $\omega + \omega$ is deserving of the symbol 2ω. And then we can begin again:

$$1,2,3,\ldots,\omega,\omega + 1,\omega + 2,\omega + 3,\ldots,2\omega,2\omega + 1,2\omega + 2,2\omega + 3,\ldots,3\omega,3\omega + 1,3\omega + 2,3\omega + 3,\ldots$$

1. Above we showed how the transfinite ordinal 2ω arose. Show how the transfinite ordinal 3ω arises.

2. Show how the transfinite ordinal 4ω arises.

3. Show how the transfinite ordinal 5ω arises.

4. Can you generalize Investigations 1-3 to show how the transfinite ordinal $n\omega$ arises where n is any natural number? Explain.

5. Now that we have the transfinite ordinals $\omega,2\omega,3\omega,4\omega,\ldots$ what is their immediate successor? Explain.

6. Explain why it is appropriate to write the transfinite ordinal just obtained as ω^2.

7. Now we begin again with ω^2 forming $\omega^2+1,\omega^2+2,\omega^2+3,\ldots$ What is the immediate successor of this collection?

8. Show how one obtains the transfinite ordinal $\omega^2 + \omega + 3$.

9. After passing by several more limiting processes, we arrive at the transfinite ordinal $2\omega^2$.

Then, after several more limiting processes $3\omega^2$. Then $4\omega^2$. And $5\omega^2$. Etc. What is the immediate successor of this collection?

10. Explain why it is appropriate to write the transfinite ordinal just obtained as ω^3.

11. If we skip even more intervening limiting processes, we can go from ω^3 to ω^4. Then to ω^5. And then to ω^6. Etc. What is the immediate successor of this collection?

12. Explain why it is appropriate to write the transfinite ordinal just obtained as ω^ω.

13. If we skip even more intervening limiting processes, we can go from ω^ω to $\omega^{2\omega}$. Then to $\omega^{3\omega}$. Then to $\omega^{4\omega}$. Etc. What is the immediate successor of this collection?

14. Explain why it is appropriate to write the transfinite ordinal just obtained as ω^{ω^2}.

15. Explain how we can form ω^{ω^ω} starting from ω^{ω^2}.

16. In Unit 1 you wrote truly large numbers using powers of 10. Using this as an analogy, describe the largest transfinite ordinal you can.

17. Can you describe the immediate successor of the ordinal just described? Explain.

Let's glimpse a more fabulous number system which transcend even the ordinals – the surreal numbers.

The Surreal Numbers

In response to the challenge of an arithmetic of transfinite ordinals, in the mid 1960's John Horton Conway (English Mathematician; 1937 -) created a new number system that dealt with these difficulties. This number system, called the **surreal numbers**, found a place for numbers such as $\omega-1$ and $\frac{1}{\omega}$. The latter is of historical import. For as described in the introduction to this chapter the "evanescent" infinitesimal had long troubled the foundations of calculus.

Yet consider the possible meaning of $\frac{1}{\omega}$. As ω succeeds all of the natural numbers we have 1 < ω, 2 < ω, 3 < ω,... But this means that $\frac{1}{\omega} < \omega$, $\frac{1}{\omega} < \frac{1}{2}$, $\frac{1}{\omega} < \frac{1}{3}$ Certainly $\frac{1}{\omega}$ should be non-negative. We wouldn't expect it to be equal to zero since there are transfinite ordinals beyond ω whose reciprocals should be smaller than $\frac{1}{\omega}$. But the numbers 1, $\frac{1}{2}$, $\frac{1}{3}$, $\ldots$ get smaller and smaller. In fact, they get indefinitely close to 0. $\frac{1}{\omega}$ is thus between 0 and points that become indefinitely close to zero. It is exactly Berkelely's "ghost of a departed quantity" in the flesh. It is not zero, but it is smaller than any real number we can name. It is an infinitesimal! Yet it is not the only one. $\frac{1}{\omega+1}$ is another, as is $\frac{1}{\omega^{\omega}}$, and etc. ; an infinity of infinitesimals trapped between 0 and the smallest of the small!

18. Explain why the term “ordinal” is an appropriate title for the ordinal numbers.

19. Explain why the term “cardinal” is an appropriate title for the cardinal numbers.

20. What does the term “surreal” mean? Why do you think Conway chose this an a name for the surreal numbers?

Use your knowledge about the relative sizes of ordinal numbers to decide which of the following surreal numbers is larger:
and.

21. $\frac{1}{\omega}$and $\frac{1}{\omega+1}$.

22. $\frac{1}{\omega}$and $\frac{1}{\omega^2}$.

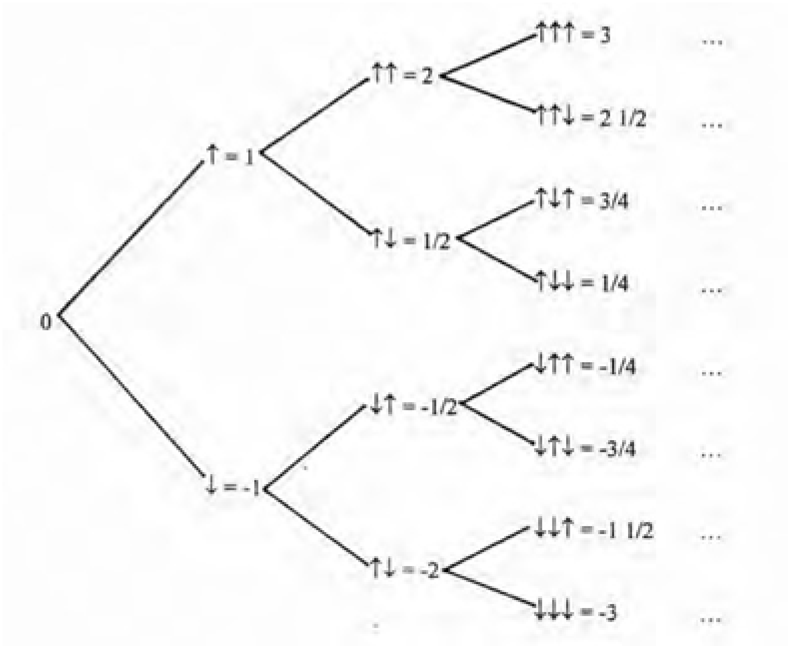
23. $3\omega - 1$ and ω^2.

24. 0 and $\frac{1}{\omega^{\omega}}$.

Surreal Diagrams

The surreal numbers are best introduced inductively, that is step by step.
We start from its most basic units: ↑ and ↓. When these symbols appear by themselves, they represent 1 and −1 respectively. But in conjunction with other occurrences of these symbols you can only determine the value of the arrows in relation to the other symbols that appear.

As we introduce the surreal numbers in this fashion it is most enlightening to organize them into a tree. The first several steps are shown below:



Following the pattern in the tree above, write each of the following real numbers as surreal numbers:

25. 4

26. $2\frac{1}{2}$

27. 9

28. $\frac{3}{8}$

29. $\frac{5}{16}$

Following the pattern in the tree above, write each of the following surreal numbers as real numbers:

30. ↑ ↑ ↓ ↑

31. ↑↓↓↓↑

32. ↑↓↑↓↑

33. ↑↓↓↓↑↑

34. Write the natural numbers 5,6,7, and 8 as surreal numbers.

35. Can you think of a way to write ω as a surreal number? Is there any way to abbreviate this notation to a reasonable size? Explain.