Extension:

Riemann's Rearrangement Theorem

Peter Lejeune Dirichlet (German Mathematician; 1805 - 1859) is the earliest person we have on record of being aware that rearranging an infinite series may change its sum. He discovered this in 1827. It was not until 1852 that the question was investigated more fully, this time by Riemann who had sought the advice of his elder Dirichlet. The rearrangement theorem that now bears his name was proven by 1853 but was published posthumously in 1866, and only then because its importance was recognized by Richard Dedekind (German mathematician; 1831 - 1916).[1]

Riemann's Rearrangment Theorem is counterintuitive, showing that infinite series do not satisfy the commutative law of addition that we are used to. But in some sense it is maximally counterintuitive, showing that a conditionally convergent series can be rearranged to converge to any value one desires! With such surprises, one might expect that proving this result would be very difficult. However, nothing could be further from the case. Rather, the direct and simple proof adds greatly to the beauty of this result.

Here we illustrate intuitively how the proof proceeds. A formal proof relies on little more than rigorous use of the key definitions of the ideas investigated here and below. Our approach follows that of Dunham in The Calculus Gallery who follows Riemann's original proof.

1. Suppose the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent. Explain why the individual terms a_n must converge to 0 in the limit as $n \to \infty$.

2. Explain how we can rearrange the infinite series $\sum_{n=1}^{\infty} a_n$, which may include both positive and negative terms, into the difference of two infinite series with all positive terms. i.e. $\sum_{n=1}^{\infty} a_n = (c_1 + c_2 + c_3 + \ldots) - (d_1 + d_2 + d_3 + \ldots$, where $0 \leq c_n, d_n$. (Note: We chose the labels c and d to symbolize the "credits" and "debits" of the original series.)

[1] See "Riemann's Rearrangement Theorem" by Stewart Galanor, *Mathematics Teacher*, vol. 80, no. 8, Nov. 1987, pp. 675-81 and A Radical Approach to Real Analysis by David Bressoud.

3. Suppose the infinite series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent. Explain why the associated infinite series of credits and debits (i.e. the series $c_1 + c_2 + c_3 + \ldots$ and $d_1 + d_2 + d_3 + \ldots$ from Investigation 2 both must diverge.

4. Pick any target value, which we denote by *T*, you would like the rearranged series to converge to.

5. Beginning with c_1 begin adding successive credits. Explain why the sum must eventually surpass *T*.

6. If you stop adding successive credits as soon as the sum surpasses T, how far from T can the sum be?

7. To the credits in Investigation 6 begin subtracting successive debits starting at d_1. Explain why the sum must eventually fall below T.

8. If you stop subtracting successive debits as soon as the sum falls below T, how far from T can the sum be?

9. Explain how you can continue this process to rearrange the series so it has the form

$$(c_1 + c_2 + \ldots c_{n1}) - (d_1 + d_2 + \ldots d_{m1}) + (c_{n1+1} + c_{n1+2} + \ldots c_{n2}) - (d_{m1+1} + d_{m1+2} + \ldots d_{m2}) + \ldots$$

with partial sums mimicking those in Investigation 6 and Investigation 8.

10. Explain why the rearrangement just created must have sum T.

Most infinite series are remarkably hard to analyze, as the quote above from Abel seems to suggest. Whether their sum is finite or not can often be ascertained, but rarely can the exact value of the sum be found. Nonetheless, many important and remarkable infinite series have been discovered, several of them in the search for methods of approximating the ubiquitous constant . The following infinite series is known as the **Gregory series**:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

11. Write out the next ten terms in this series.

12. Sum the first five terms in this series.

13. Sum the first ten terms in this series.

14. Show that your answers to the previous problems are close to $\frac{\pi}{4}$.

Although it is hard to demonstrate, the Gregory series does has as a sum $\frac{\pi}{4}$. That is

a fact that was known not only by Gregory (Scottish Mathematician and Astronomer; 1638 - 1675), but Leibniz (German Mathematician and Philosopher;1646 - 1716).
Almost two centuries earlier than any of these well-known mathematicians, the Indian mathematician Madhava of Sangamagrama (1350- 1410) published his discovery of this infinite series. Perhaps we should rename the series!